Applied Mathematics 202:
Physical Mathematics II

FAS Course web page: http://isites.harvard.edu/k104109 *(Spring 2015)*

Last updated: April 26, 2015.
Feel free to write call or visit us with any questions.

1 Administrative

Instructor: Eli Tziperman (eli at seas.harvard.edu).

TFs: see course web page.

Day & time: MWF 10-11

Location: MD G125

Sections: once a week, see FAS course web page for times and places.

1st meeting: TBA

Office hours: Eli: Please see course web page for times; 24 Oxford, museum building, 4th floor, room 456. TFs: please see course web page.

Textbooks: Page or section numbers from the relevant textbook for any given lecture are given in the detailed syllabus below. The main course textbook is

CP Carrier & Pearson, *Partial Differential Equations, theory & technique*

Also used at times,

Za Zauderer, *Partial differential equations of applied mathematics*, 3rd edition, Wiley.

Gr Greenberg, *Advanced Engineering Mathematics*, 2nd edition.

Gr2 Greenberg *Foundations of Applied Mathematics*

HW Haltiner and Williams, *Numerical prediction and dynamic meteorology*, 2nd edition, (Wiley, 1980).

OC Ockendon, Howison, Lacey & Movchan. *Applied partial differential equations*, (Oxford, 2003)

RM Richtmyer and Morton, *Difference methods for initial value problems*, (Interscience/Wiley 1967)

Ze Zemyan, *The classical theory of integral equations*, Springer 2012.

Wh Whitham, *Linear and nonlinear waves*, 1974.

Ha Haberman, *Applied partial differential equations with Fourier series and boundary value problems*. 4th edition, 2003.

CG Cross and Greenside, *Pattern formation and dynamics in nonequilibrium systems*, Cambridge University Press New York, 2009.

Some additional ones,

Hi Hildebrand, *Advanced Calculus for Applications* (2nd Edition).

CH Courant & Hilbert, *Methods of Mathematical Physics*, vols. 1,2 (Wiley, 1989)

MF Morse & Feshbach, *Methods of theoretical physics*, vols. 1,2 (MGH, 1953)

Supplementary materials: Additional materials from several additional textbooks and other sources, including Matlab programs used in class, may be found here. Follow links below for the specific source material for each lecture. In order to access these materials from outside the Harvard campus or via the wireless network on campus, you'll need to connect via the web Harvard VPN client or use the VPN software which can be downloaded from the FAS software download site.

Prerequisites: Applied Mathematics 105 (intro to ODEs and PDEs) or equivalent; also useful: Applied Mathematics 104 (complex analysis) or equivalent. *Note:* Applied Mathematics 201 and 202 are independent courses, and may be taken in any order.

Sections: The TFs will hold weekly sections and have weekly office hours. During the sections, they will discuss and expand on the lecture material and solve additional example problems.

Programming Skills: some of the demonstrations and homework assignments will be Matlab-based. Some previous programming exposure is assumed, although not necessarily in Matlab. Students are asked to download and install Matlab on their computers from the FAS software download site.

Homework: homework will be assigned every nine to ten days, and will be due nine to ten days later in class. The homework assignments are essential for understanding the lecture material and introduce you to some important extensions. We strongly encourage you to discuss and work on homework problems with other students (and with the teaching staff, of course), but you should ensure that any written work you submit for evaluation reflects your own work and your own understanding of the topic, and is written in your own words.

Grading: Homework: 70%; midterm (during evening hours, Wednesday March 11, 2015, 7-9pm) 15%; final 15%.

Readings: Students are encouraged to read from the sources pointed to by this extended syllabus, to obtain a broader perspective on the course material.

This document: http://www.seas.harvard.edu/climate/eli/Courses/APM202/2015spring/detailed-syllabus-apm202.pdf, available from within campus or via VPN.

Contents

2 Outline

Theory and techniques for finding exact and approximate analytical solutions of partial differential equations: eigenfunction expansions, Green functions, variational calculus, transform techniques, perturbation methods, characteristics, selected nonlinear PDEs, introduction to numerical methods.
Note: Applied mathematics 201 and Applied mathematics 202 are independent of each other and may be taken at any order.
Prerequisites: Applied Mathematics 105 or equivalent; (also useful but not required: Applied Mathematics 104 or equivalent).

3 Syllabus

Follow links to see the source materials, including Matlab demo programs used for each lecture.

3.1 Introduction, overview

downloads.
Logistics, course requirements, textbooks, overview of the course, what to expect and what not to expect.

3.2 Separation of variables and basics

for diffusion, wave and Laplace equations. downloads.

1. Diffusion:

   (a) Motivation, derivation of diffusion-advection eqn (notes, section 2), scaling of diffusion-advection (same notes, section 3). Separation of variables for a basic case (**Gr**§18.3, example 1, pp 954-958; then Matlab demo diffusion_1d_SL.m).

   (b) Some generalities and complications: Sturm-Liouville expansions: S-L theorem (**Gr**§17.7.1 p 887-888 until but not including example 1; inner product definition eqn 14 p 890; theorem 17.7.1 p 891). Examples: the diffusion equation we solved above, leading to Fourier series.

   (c) More interesting cases: periodic and singular S-L (**Gr** p 906, eqns 2-5); examples of singular S-L: (i) diffusion on a disk (**Gr**§18.3, example 5, pp 968-971) leading to Bessel eqn (example 2 p 908-909, eqns 12-24); (ii) diffusion on a sphere (notes-diffusion-1d-sphere-using-Legendre-SL.pdf, only up to derivation of Legendre equation for x structure on second page) leading to Legendre eqn (example 2, p 910, eqns 25-28).

   (d) Integrating factor to bring to S-L form (notes)

   (e) Nonhomogeneous problems via eigenfunction expansion (**CP**§1.5, pp 13-16)

   (f) Nonhomogeneous, time dependent, boundary conditions (**CP**§1.7, pp 17-18)

   (g) Similarity solution (notes). [Time permitting:] Kelvin's wrong estimate of the age of the Earth (England et al, eqns 1-4); maximum principle (**CP**§1.9 p 19-20);

2. Wave: Motivation and derivation (**Gr**§19.1 p 1017-1019, eqns 2,5-9). Finite domain, separation of variables and S-L, using a 2d vibrating membrane example to keep it different from the 1d diffusion case above (**Gr**§19.3 p 1035-1039). Factoring into 1st order and D'Alembert's solution, e.g., for an infinite domain (**CP**§3.3 p 38-41), semi-infinite domain using the method of images (**CP**, problem 3.4.3 p 43). Using energy integral to demonstrate uniqueness (**CP**§3.11 p 52-53).

3. Laplace equation:

   (a) Motivation, derivation (notes, section 4).

(b) Simplest problem on a rectangular, using superposition principle in case b.c need to be expanded on more than one boundary (**Gr**§20.2, pp 1059-1063)

(c) Separation of variables and S-L, for a 2d problem on a cylinder using non-Cartesian coordinates (**CP**§4.4, p 63-64 eqns 4.9-4.12);

(d) Average value principle derived from 4.12, and the resulting minimum and maximum principles (also, not essential: a more general discussion is given in **CP**§4.3).

(e) Integral constraints: consistency requirement with Neumann b.c. $\nabla^2 u = f(x,y)$. Mathematically: integrate equation over the domain; physically: prescribed heat source $f(x,y)$ plus prescribed heat flux into the domain must sum to zero for a steady state solution to the diffusion equation to be possible (**Gr** Exercise 17, p 1069).

(f) Smoothing of discontinuities in boundary conditions (**CP**, problem 4.8.9 p 74, see also last paragraph starting on page 67 and notes).

(g) Real and imaginary parts of an analytic function are harmonic; complex variable and conformal mapping (**CP**§4.7 p 68-71).

(h) Some problem are non-separable due to eqn ($u_{xx} + u_{xy} + u_{yy} = 0$) or b.c.

3.3 1st order PDEs and characteristics

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(a) **Motivation:** advection (notes, section 2); traffic as an example of conservation laws (Haberman, p 549, eqns 12.6.10 and 12.6.12, with $c(\rho) \equiv dq(\rho)/d\rho$); age distribution (population at age a and time t is equal to population at age $a - dt$ and time $t - dt$ minus those that died between $a - dt$ and a:
$n(a,t) = n(a-dt, t-dt)(1 - \mu(a-dt)dt)$; this implies
$[n(a,t) - n(a-dt, t-dt)]/dt = -\mu(a-dt)n(a-dt, t-dt)$; use
$n(a-dt, t-dt) = n(a,t) - \partial_t n\, dt - \partial_a n\, dt$ to finally get $\frac{\partial n}{\partial t} + \frac{\partial n}{\partial a} = -\mu n$); factoring 2nd order wave PDE into first order PDEs.

(b) **Linear** 1st order PDEs in two independent variables: solution using characteristics. Examples: (1) unidirectional wave equation (section 1 in notes, based on **Za**, example 2.2 p68-69); (2) a slightly more advanced case (see section 2 of notes, based on **CP**§6.1, eqns 6.6-6.10 p 93-94, plus problem 6.2.1, pp 93-94).

(c) **Quasilinear, shocks:** quasilinear 1st order PDEs in two independent variables, characteristics. (1) Equations (6.13-6.15) in **CP**§6.3p 97 (section 3 in notes). (2) Traffic problem, demonstrating multivalued solutions and shock formation, shock conditions, velocity of shock propagation (section 4 in notes, and then Haberman, §12.6.3-12.6.4, p 549-555). (3) An example showing multivalued solution and the formation of a shock (section 5 of notes) and Matlab program. Finding the location of the shock in the case of multivalued solution using Whitham's equal-area method (Haberman p 557 Fig. 12.6.12).

(d) A bit of **theory**: characteristics here have the same meaning as with 2nd order PDEs: specifying the value of the function on characteristic curves does not allow us to solve near these curves (heuristically: because signal propagates along the curves rather than away from them. See details in **CP**§6.5 p 99-101). Implication: solvability and Jacobian of characteristic curves and curve along which initial values are specified (Jacobian vanishing means they are parallel and the problem is not well posed); *Time permitting:* extension to more independent variables (**CP**§6.7, eqns 6.18-end); envelope of characteristics (**CP**§13.1).

(e) **Nonlinear** 1st order PDEs, characteristics, Charpit's equations; example: Eikonal (Eiconal) equation (notes from Oxford course by Dr. Norbury; see also, although less transparent, **Za**§2.4 pp 102-111 or **OC**§8.2.1, §8.2.3 pp 360-368).

3.4 Classification of 2nd order PDEs

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(a) Intro: calculating the solution from Cauchy data (specified value and normal derivative) along the y axis, and expanding the solution as $\phi = \phi(0,y) + x\phi_x(0,y) + \frac{x^2}{2!}\phi_{xx}(0,y) + \cdots$; Such an expansion is possible according to the (not so useful, see below) Cauchy-Kowalewski theorem. Discuss when this expansion might fail in the above example (**CP**§5.1 p 75-76). More generally, characteristics of a 2nd order PDE are lines along which such Cauchy data are insufficient for solving for the function.

(b) Next, trying to find a local solution based on Cauchy data on an arbitrary curve using a power expansion. Switch from (x,y) to curves (ξ,η); note: if curves $\xi(x,y)$, $\eta(x,y)$ are parallel at a point, the cross product of their gradients should vanish, so that $0 = \nabla\xi \times \nabla\eta = \xi_x\eta_y - \xi_y\eta_x = J(\xi,\eta)$; Eq'n takes the form $\phi_{\xi\xi}(A\xi_x^2 + 2B\xi_x\xi_y + C\xi_y^2) + \ldots$ (**CP**§5.2). This cannot be solved for if the coefficient of $\phi_{\xi\xi}$ vanishes, and the condition that it does leads to a quadratic equation for ξ_x/ξ_y (or ξ_y/ξ_x if $\xi_y = 0$).

(c) Case I: hyperbolic (wave, two real characteristics, $B^2 - AC > 0$): when this condition is satisfied at a given point, there are two solutions for $\xi_x/\xi_y = -dy/dx$ and therefore two curves $y(x)$ that pass through this point, along which the coefficient of $\phi_{\xi\xi}$ vanishes. Cauchy's data are therefore insufficient for solving for ϕ near these two curves. (CP§5.4, p 79-80, only until and including eqn 5.8). Example: for the standard wave equation, $\phi_{tt} - c^2\phi_{yy} = 0$ we have $A = 1$, $C = -c^2$, $B = D = E = F = G = 0$, so that $\pm\sqrt{B^2 - AC} = \pm c$. Comments: (1) the slope of the line $y(x)$ along which $\xi(x,y) = \text{const}$ is determined as follows, $0 = \frac{d}{dx}\text{const} = d\xi/dx = (\partial_x\xi) + (\partial_y\xi)dy/dx$, which implies $dy/dx = -\xi_x/\xi_y$. (2) Note the similarity of eqn 5.8 to the one derived in D'Alembert's solution, $\phi_{\eta\xi} = 0$ by using $\xi = x + ct$, $\eta = x - ct$ for the simple wave equation $\phi_{tt} = c^2\phi_{xx}$. (3) Physical interpretation: the two characteristics correspond to the trajectories of wave propagation in the (x,t) plane. Because the value of ϕ is effectively propagated along these curves, it cannot propagate off the curves in order to allow us to find the solution away from the curves. With any other curve on which Cauchy data are prescribed, these data can be carried off the curve by wave propagation to allow calculation of the solution there. (4) This is an important lesson on a consistent b.c. formulation for the wave equation and on what to avoid.

(d) Case II: parabolic ($B^2 - AC = 0$, one characteristic, **CP**§5.5, p 81), e.g., standard 1d diffusion eqn $\kappa\phi_{xx} - \phi_t = 0$, where $A = \kappa$, $B = 0$, $C = 0$, implying that the characteristics in the (x,t) plane satisfy $dt/dx = -B/A = 0$ so that they are given by $t =$ const and are parallel to the x axis. Note that we cannot specify arbitrary Cauchy data along a characteristic in this case because if we specify ϕ along $t = 0$, ϕ_{xx} is calculated by differentiating ϕ, and ϕ_t is then determined from ϕ_{xx} via the equation itself. Thus the normal derivative, ϕ_t, is determined from the function along the characteristic. Therefore in this case the characteristic is a curve along which *arbitrary* Cauchy data *cannot* be specified rather than that they are *insufficient* for solving around the curve. While the classification indicates that we cannot solve the equation by a power series method with Cauchy data specified on the characteristic, we can still find a solution via time integration as long as the Cauchy data are consistently specified. Cauchy-Kowalewski theorem is clearly not so relevant in this case.

(e) Case III: elliptic (Laplace, $B^2 - AC < 0$, no real characteristics): can always find a solution near a curve with specified Cauchy data via a power series expansion. Example: Laplace equation, $\phi_{xx} + \phi_{yy} = 0$, $A = 1$, $C = 1$, $B = D = E = F = G = 0$, $B^2 - AC = -1$. Physically, think steady diffusion, and note that information diffuses to the neighborhood of the curve, allowing the solution to be determined. However, one can come up with examples where the solution does not depend continuously on the Cauchy data (**CP**§5.10, p 90, middle paragraph), so that Cauchy-Kowalewski theorem is again not useful from practical point of view. Still, one can show that any elliptic equation may be locally transformed to the Laplace equation, providing some insight.

(f) *Time permitting:* Characteristics as only possible location of discontinuities in second derivatives, as paths for signal propagation, as curves where Cauchy data cannot be specified arbitrarily. (**CP**§5.8).

3.5 Transform methods

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used to reduce the order of the PDE or even convert it into an ODE. Fourier transforming the spatial dimension of a PDE often leads to a solvable ODE initial value problem in the time dimension; solving it and inverse Fourier transforming solves the original problem. Similarly, Laplace transforming a wave or diffusion equation leads to a boundary value problem, which can then be solved and inverse Laplace transformed.

(a) The **Fourier transform** is especially useful in problems with infinite spatial dimension(s). Fourier transform and inverse formula in n dimensions, (proof),

$$F(k) = \frac{1}{(\sqrt{2\pi})^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}) d\mathbf{x}; \quad f(x) = \frac{1}{(\sqrt{2\pi})^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-i\mathbf{k}\cdot\mathbf{x}} F(\mathbf{k}) d\mathbf{k},$$

see also basic 1d definition and properties in **Za**§5.2, eqns 5.2.6-7, 5.2.10-11 p 256-257. Application to *diffusion* equation (**Za** example 5.2, p 261-262, eqns 5.2.26-5.2.39); *wave* equation, and derivation of D'Alembert's solution (**Za** example 5.3, p 264; is there a missing minus sign in front of one of the $G(\lambda)$ integrals in 5.2.46?); *Laplace* equation on a half-plane (**Za** example 5.4, eqns 5.2.50-5.2.58, p 265 to line 4 on p 266).

(b) Higher dimensional Fourier transforms (**Za**§5.4, eqns 5.4.1-2, p 281). Example: 2d Modified Helmholtz equation (**Za** pp 287-288, eqns 5.4.26-5.4.36, skip details in eqns 5.4.32-5.4.35).

(c) The **Laplace transform:** is especially useful for initial value problems. The transform and inverse transform are,

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt; \quad f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s)\, ds.$$

Transform of time derivatives and other basic properties (**Za**§5.6, pp 297-299). In particular the useful property for IVPs $\mathcal{L}[f^{(n)}] = s^n \mathcal{L}[f] - s^{n-1} f(0) - \ldots - s^0 f^{(n-1)}(0)$. Applications: *diffusion* equation in a semi-infinite domain, noting infinite propagation speed of diffusing signal (**CP**§2.1, pp 24-25); *wave* equation (statement of problem in **CP**§3.3, eqn 3.5 on p 39; then solution method in **CP**§3.9, pp 50-51, with details given in the notes; and the final solution is given by **CP**§3.3, eqn 3.7 on p 40).

(d) *Time permitting:* Fourier sine and cosine transforms; Hankel transform (**Za**§5.5).

3.6 Green's functions

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(a) Motivation, using Green's function for nonhomogeneous problems, and/ or to satisfy initial and boundary conditions. Solving once for Green's function allows to easily solve many nonhomogeneous cases. We'll demonstrate several methods of solving for Green's function.

(b) δ function reminder (viewed either as a generalized function defined only through its integrals when multiplying other functions: **Za**, p 438-441; or, alternatively, viewed as a limiting case of well behaved functions that tend to a delta function as a parameter goes to zero or infinity: **CP**, §2.5 p 30-33 for 1d case and p 141, eqns 9.5-9.8 for 3d case)

(c) Laplace equation in a finite domain:

i. Construction of equation *and* boundary conditions for Green's function for Laplace equation using the divergence theorem; (section 1 in notes; and then the somewhat more general case in **Za**§7.1 pp 410-412. Equivalently: **CP**§9.2, pp 142-143).

ii. Solving for the Green's function, Method 1: using eigenfunction expansion, useful for self-adjoint operators in bounded domains (first, theoretical derivation in **Za**§3 p 444 eqns 7.3.1-7.3.6 plus my notes that elaborate on the derivation of the eigenfunction expansion and show that the expansion indeed leads to a green's function; then example of a Laplace equation on a rectangle, **Za** example 7.6, eqns 7.3.26-7.3.41 pp 448-450); *Time permitting:* modification for Newman conditions (**Za** example 7.7, p 454-456);

iii. Method 2: finding an explicit solution for the Green's function using jump conditions at $x=\xi$. Start with a general derivation (**Za** eqns 7.3.7-7.3.12, pp 445-446); proceed with an example for an ODE (**Za** example 7.5, eqns 7.3.13-7.3.18, pp 446-447), and finish general discussion (**Za** eqn 7.3.1, p 447). Example: elliptic eqn on a rectangle again, this time using this alternative method (**Za** eqns 7.3.42-7.3.45, pp 450-451).

iv. Method 3: Finding a Green function solution g_1 with the right singularity at $x-\xi$, and adding a harmonic function p satisfying Laplace equation in interior and b.c. such that $g=g_1+p$ satisfies the needed Green's function b.c. First, note that (in 3d!) $\Delta(1/r)=\delta(\vec{x}-\vec{\xi})$, [in 2d, the equivalent eqn is $\Delta \log r=\delta(\vec{x}-\vec{\xi})$] where $r=|\vec{x}-\vec{\xi}|$ (**CP** p 141, first eqn and then 9.7-9.8). Next, definition of two components of Green's function, $1/r$ and p and an example in a sphere (**CP**§9.2 pp 142-144). This is based on Kelvin's inversion formula for a solution of Laplace eqn outside of a sphere in terms of a solution inside the sphere, briefly explained in **CP** question 7.2.5 on p 109); *Time permitting:* Modified Laplacian (**CP**§9.6, pp 152-154, uses modified Bessel functions K_0, I_0).

(d) Diffusion equation in an unbounded domain. Start by deriving the eqn and b.c. for the Green's function using section 3 in these notes. In order to solve for the Green's function, consider Method 4: using Fourier transform, useful in unbounded domains (**Za**§7.4, eqns 7.4.1-7.4.6, pp 462-463). *Time permitting:* see also here for solving for the Green's function of the wave equation using Fourier transform.

(e) Using the new insight that to derive Green's function we want to consider the integral based on the adjoint operator, $\int(L[u]w-L^*[w]u)$: first derive again eqn and b.c for Poisson's eqn $\nabla^2 u=f(x,y)$ with some b.c; then consider the elliptic equation with non constant coefficients from **CP** HW problem 9.7.4 p 155.

(f) Wave equation: can use several of the previous methods, but consider an alternative, Method 5: using Laplace transform (**CP**§9.8, pp 156-157 eqns 9.37-9.40, with an example of a moving wave source; the actual derivation of the Green's function solution (9.38) is too brief, so see section 2 in notes). Show image of the moving flame solution from the course web page. *Time permitting:* for an alternative approach using both Fourier and Laplace transforms, see eqns 18-37 on pp 3-5 of these notes from here.

3.7 Variational methods

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(a) Motivation: minimum area of soap films (derivation of problem using notes); equivalence of variational problems and PDEs (**CP** problem 10.2.7, p 166, but leaving details for later); data assimilation and weather prediction (notes); access to additional numerical methods.

(b) Introductory example, calculus of variations, demonstrated first by differentiating with respect to epsilon and then using the δ notation (**CP**§10.1, pp 161-164). Demonstrate delta for simple objects first, e.g., $\delta(u^2) \equiv (u+\delta u)^2 - u^2 \approx 2u\delta u$.

(c) Another simple example, e.g., $\delta I = \delta \int (xuu_x^2 + yuu_y^2 - u^2)dA = 0$ with $u = 0$ on boundaries.

(d) A variational principle from squaring a given PDE: show explicitly **CP** problem 10.2.7, p 166, but using my notes.

(e) Natural boundary conditions (**CP**§10.3, p 167, eqns 10.4-10.7 plus the paragraph after this last equation).

(f) Another example of natural boundary conditions: $I = \int_A (xu_x u_y + \frac{1}{2}u^2)dA + \int_\Gamma (fu)ds$, for the unit square, $0 \le (x,y) \le 1$, work out the details of boundary conditions on each edge separately.

(g) Constraints and adjoint equations:

   i. Reminder of Lagrange multipliers (**CP**§10.4, pp 168 and then text around eqn 10.12).

   ii. An actual variational problem, posed at the beginning of the last paragraph ("we now extend...") on p 170, and then solved on p 171, eqn 10.14 to end of page.

   iii. Note that when constraint is applied at each point, Lagrange multiplier becomes a function.

   iv. A time dependent example: calculating sensitivity efficiently using adjoint equations (notes).

   v. Interpretation of the adjoint variable, and the fact that it is solved using backward integration in time.

   vi. (time permitting) Adjoint of finite difference from Laure Zanna's notes, sections 4, 5, 6

vii. Adjoint of finite difference vs finite difference of adjoint: (i) Derive continuous adjoint for $0 = \delta\mathcal{L} = \delta\int \frac{1}{2}c^2 + \lambda(x)(c_{xx} - \kappa c)dx$ with b.c of $c_x = 0$ at $x = 0, L$. (ii) Derive finite difference adjoint from finite difference Lagrange function $\sum_i c_i^2 + \lambda_i(c_{i+1} - 2c_i + c_{i-1}0 - \kappa\Delta x^2 c_i)$, first without worrying about b.c. (iii) introduce treatment of b.c via writing $c_{xx} \approx (c_x|_{x+\Delta x/2} - c_x|_{x-\Delta x/2})/\Delta x \approx ((c_{i+1} - c_i) - (c_i - c_{i-1}))/\Delta x^2$ and using $c_x = 0$ at boundaries. Write the Lagrange function for this case, no need to derive equation and b.c. for adjoint variable. (iv) Advantages of adjoint of finite difference: gradient of finite difference adjoint is accurate to round off error, while it is $O(\Delta x)$ for finite difference of adjoint. (v) Problems with adjoint of finite difference if scheme is not carefully chosen: possible numerical signal that may lead to numerical instabilities; show two figures from Sirkes paper.

(h) Approximations: Rayleigh-Ritz method (**CP**§10.6, pp 174-177, until the end of the paragraph following eqn 10.23). A brief reminder: finite element (**CP**§10.8) is closely related to Rayleigh-Ritz method.

3.8 Perturbation methods

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(a) Motivation.

(b) Scaling, non-dimensionalization. Example 1: scaling estimate for maximum height of a thrown ball (notes, "idea 2" on page 2). Example 2: forced pendulum: $m\ddot{x} + r\dot{x} + kx = F\sin\omega t$, use $x = Lx'$, $t = Tt'$, to get $\ddot{x}' + (T^2 r/Tm)\dot{x}' + (T^2 k/m)x' = (T^2 F/mL)\sin\omega T t'$. Assuming this is essentially a harmonic oscillator with weak forcing and weak dissipation, *choose* $T = \sqrt{m/k}$ (natural frequency), to find $\ddot{x}' + (r/\sqrt{mk})\dot{x}' + x' = (F/kL)\sin\omega\sqrt{m/k}t'$, and let $\varepsilon = r/\sqrt{mk} \ll 1$. For the forcing to be small, *choose* again $F/(kL) = \varepsilon f$, where f is a nondimensional, $O(1)$, forcing amplitude. Define $\omega' = \omega/\sqrt{k/m}$ to finally get $\ddot{x}' + \varepsilon\dot{x}' + x' = \varepsilon f \sin\omega' t'$. Example 3: rotating shallow-water fluid dynamics (notes sections 1-2).

(c) Regular perturbations: Example 1 (**CP**§8.1, pp 127-129; order: eqn 8.1 then 8.4, 8.5 and the following equations and b.c., then solution for $\phi^{(1)} = \psi^{(1)} - \psi^{(0)}$ from 8.2 and last eqn on p 127; then write $\psi^{(2)} = \phi^{(0)} + \phi^{(1)} + \phi^{(2)}$ from p 128 after eqn 8.3).

(d) Singular perturbations and boundary layers: First, ODE reminder (**CP**§15.1, use these notes, section 1). Then, Example 1: PDE case from (section 2 of notes, based on **CP**§15.1). Example 2: quasi-geostrophy again, steady large-scale ocean circulation, with Gulf Stream as a boundary layer (notes, section 5). Example 3: Boundary (initial) layer in a singular perturbation wave problem (**Za**, example 9.10, pp 606-610). Example 4: degeneracy of geostrophy and quasi-geostrophic shallow water vorticity equation (notes sections 3-4).

(e) Boundary perturbations: when the boundary shape is close to that of a simpler object (e.g., circle) and the difference may be treated as a perturbation (**CP**§8.4). Another example: surface gravity waves, where the b.c is applied at an approximate boundary location rather than the accurate one (my hand-written notes, based on Knauss, introduction to physical oceanography, 2nd edition, 1996m pp 195-198).

(f) A brief reminder of 1d bifurcations: saddle node and pitchfork.

(g) Method of multiple scales: is again an approach to dealing with perturbations problems in which regular perturbation fails. This may happen for example in a time-dependent problem when the regular perturbation holds only for a short time due to the appearance of secular terms in the higher order perturbation terms which lead to a failure of the perturbation expansion at longer times.

  i. ODE example: a weakly nonlinear damped oscillator. (**Ha**§14.9.1, pp 696-699)

  ii. PDE example: a weakly unstable nonlinear diffusion equation. This is an example that occurs in the study of pattern formation (**Ha**§14.9.3, pp 703-705).

3.9 Integral equations

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(a) Motivation and examples: random walks, radiative transfer (notes); Integral transforms are an example of an integral equation (e.g., Laplace transform $F(s) = \int_a^\infty e^{-st} f(t)dt$ is an inhomogeneous Fredholm integral eqn of the first kind for $f(t)$ in terms of the assumed known $F(s)$ with the symmetric kernel $K(x,t) = e^{-st}$); Conversion of 2nd order ODE to a Volterra integral equation of the second kind (Shestopalov and Smirnov, 2002, section 4.1 including example 5, p 8-10, eqns 31-46).

(b) Classification:

- Fredholm integral equation of the first kind: $\varphi(z) = \int_a^b K(z,z_0)\psi(z_0)dz_0$,
- and of the second kind: $\psi(z) = \varphi(z) + \lambda \int_a^b K(z,z_0)\psi(z_0)dz_0$,
- Volterra of the first kind: $\varphi(z) = \int_a^z K(z,z_0)\psi(z_0)dz_0$,
- and of the second kind: $\psi(z) = \varphi(z) + \int_a^z K(z,z_0)\psi(z_0)dz_0$.
- Homogeneous ($\varphi = 0$) and non-homogeneous; Writing a Volterra equation in Fredholm form; Volterra equations for Green's function.

(c) Fredholdm second kind for separable Kernels: definition of problem (**Ze**§1, p 1); parallel with matrix eqn $\vec{\varphi} = \vec{f} + \lambda \mathsf{K}\vec{\varphi}$; separable kernel example (**Ze**§1.2, p 4-11); summary combined Fredholm theorem (**Ze**§1.3.4 p 23-24). Discuss the condition for solvability when λ is an eigenvalue of the Kernel using a matrix example from my hand written notes.

(d) Fredholdm second kind for general kernels.

i. Method of successive substitution (**Ze**§2.2, p 33-34 to eqn 2.1); mention that this works only if $|K(x,t)| < M$ for all x,t in domain and $|\lambda| M(b-a) < 1$. Example ((a) on page 37?), make sure resolvent can be summed in close form.

ii. Method of successive approximation (**Ze**§2.3, p 37-39, to eqn 2.4; then theorem 2.3.1, p 41 until the expression for the resolvent kernel; note the different convergence condition obtained by this method although solution for resolvent is essentially the same).

iii. Example: **Ze**, p 42-43: example 1, and then example 2 (second approach, skipping the one based on the kernel being separable).

(e) Briefly: Fredholdm second kind, Hermitian Kernels. A Kernel is Hermitian if $K^*(x,t) \equiv \overline{K(t,x)} = K(x,t)$. E.g., $K = (x+t)$, $K = e^{i(x-t)}$ are Hermitian. Parallels with the matrix problem $\vec{\varphi} = \vec{f} + \lambda \mathsf{K} \vec{\varphi}$ when K is symmetric, $\mathsf{K} = \mathsf{K}^T$ (use my notes); A Hermitian Kernel K has specific properties involving its eigenfunctions and eigenvalues, and both it and its resolvent can be expanded in terms of these and the integral equation be explicitly solved giving exactly the same expression as in the matrix problem (**Ze**§3, selected theorems and then Example 2, p 116-117).

(f) Singular integral equations (infinite domain), consider only one simple but important example, of equations that can be solved using the convolution theorem and Fourier transform (**Ze**§7.2.2, p 256 until the third displayed equation on p 257).

(g) Volterra equations: convolution type eqns can be solved using Laplace transform (**Ze**§4.1 p 152 and then **Ze**§4.3 p 167-168, only eqns 4.3, 4.5 and first displayed equation on p 168). Show example 1, p 168. Mention that successive approximation and substitution etc can also be used.

3.10 Nonlinear PDEs

downloads.

So far we deal with nonlinearities in the discussion of 1st order PDEs using the method of characteristics, as well as in the case of weakly nonlinear PDEs using the perturbation methods. We now consider two classes of higher order, strongly nonlinear PDEs as a brief introduction to the very rich area of nonlinear PDEs, one leading to solitons and the other to pattern formation.

3.10.1 Solitons

downloads.

(a) Kortewegde Vries (KdV) eqn $\eta_t + c_0(1 + \frac{3}{2h_0}\eta)\eta_x + \gamma\eta_{xxx} = 0$ or $\eta_t + \eta_x + 6\eta\eta_x + \eta_{xxx} = 0$ (**Wh**, p 463, eqn 99);

(b) Linear analysis with nonlinear term set to zero and dispersion relation ($\eta = e^{i(kx-\omega t)}$, so that $\omega = -k(1+k^2)$, $c_{\text{ph}} = \omega/k = -(1+k^2)$; so that linear wave propagation is to the left, short waves propagate faster);

(c) Derivation of nonlinear propagating solutions (Karigiannis thesis, p. 3, or **Wh**§13.12, pp 467-468). Show nonlinear and linear single soliton solutions using Matlab program `mkdvB.m` on course web page (see comment in program with command line format for possible example cases). Explanation: nonlinearity exactly balances dispersion.

(d) Numerically only: elastic interaction of solutions, show two interacting solitons using same Matlab code (hence the name Solitons, indicating particle-like solutions). Note phase shift during interaction, easier to note for slower soliton.

(e) (Time permitting:) infinite number of conservation laws, inverse scattering transform (**OC**§9.7, pp 417-426, or **Wh**§17).

3.10.2 Pattern formation

downloads.

(a) Motivation: patterns in nature, pattern formation in Swift-Hohenberg model, see here (use Firefox); alternatively, using Matlab code from here, also posted here under course web page.

(b) The Swift-Hohenberg equation in 1d and 2d. Potential dynamics of Swift-Hohenberg eqn (**CG**§5.1, p 179-182).

(c) Linear stability analysis: while a uniform state leads to the global minimum value of the potential, it is unstable to small perturbations (which increase the potential) and the system then evolves to a different, local, minimum of the potential (**CG**§2.2, 2.3; Figs 2.7, 2.8, 2.9 in §2.5).

(d) Complex amplitude, phase and translation symmetry, heuristically deduced nonlinear cubic form of amplitude equation (**CG**§4.1.1).

(e) (Time permitting:) Nonlinear saturation, bifurcation theory, stripe states (**CG**§4.1.2, 4.1.3 pp 134-139).

(f) Amplitude equation, derivation using multiple-scale perturbation theory (**CG**§A2.3.2 p 512-514).

(g) Applications of amplitude equation:

   i. effect of pattern-suppressing lateral boundaries, first a single boundary and then two (**CG**§6.4.1, pp 226-229).

   ii. Eckhaus instability, (**CG**§6.4.2 p 230-233). Mention connection to Floquet theory: consider the linearized stability of a periodic orbit leading to the equation $\dot{\mathbf{x}} = A(t)\mathbf{x}$ where the matrix function $A(t+T) = A(t)$ is periodic and $\mathbf{x}(t)$ is the perturbation vector; Floquet theory tells us there is a solution of the form $\mathbf{x}(t) = \mathbf{p}(t)e^{\mu t}$ where $\mathbf{p}(t+T) = \mathbf{p}(t)$ is periodic; μ is also related to the eigenvalues of the linearized dynamics of a Poincare section of the periodic orbit (Strogatz).

iii. Existence of front solutions in a nonlinear diffusion equation and super-critical vs sub-critical pitchfork bifurcation (**CG**§8.3 p 296-299).

iv. (Time permitting:) Wavenumber selection (**CG**§8.3 p 296-308).

(h) Mention briefly and show two figures of solution of Kuramoto-Sivashinsky eqn (**CG**§5.4 p 196-199).

(i) (Time permitting:) Stability balloons, Zigzag instability of Swift-Hohenberg model (**CG**§4.2.1 p 139-147).

(j) (Time permitting:) complex Ginzburg-Landau eqn (**CG**§5.5 p 196-199).

(k) (Time permitting:) Nonlinear reaction-diffusion equations: fronts, linearized stability analysis via normal-mode eigenfunction expansion (Wikipedia; **CG**§5.6 and chapter 11).

3.11 More (Time permitting)

downloads.

(a) Ray tracing solution of wave equations; WKB calculation of amplitude variations

(b) Floquet theory

(c) Similarity solutions, Lie groups (**OC**§6.5 p 262-271).

(d) Barotropic instability, **CP** p 138, question 8.5.12).

(e) Stochastic PDEs

3.12 Review

downloads.

3.13 Brief intro to numerics

downloads.
Outline and sources for lectures by Madeline Miller: here

(a) Motivation: On the need to solve numerically: (i) even very simple equations cannot be solved analytically (e.g., chaotic systems such as the Lorenz equations); (ii) a complicated analytic solution is not necessarily more insightful than a numerical one;

in both cases one needs to plot the solution as function of different parameters to obtain understanding; (iii) a numerical solution is often the first step toward an analytic approximation which can provide interesting insights, by allowing to find which terms are dominant and which may be neglected.

(b) Basics: first order one-sided ($O(\Delta)$) vs second order center difference ($O(\Delta^2)$) approximations (in space or time; **HW** p 109) as examples of the order (accuracy) of numerical schemes; first derivative in time (e.g., for diffusion equation) using first order Euler forward and improved Euler-Forward (**St** pp32-33).

(c) Numerical stability: definition of convergence and stability (**HW**, p 122); stability analysis using the matrix method (**HW**, §5-5-1, pp 123-126)); Stability analysis using the Von Newman method (**HW**, §5-5-2, pp 127-129; Euler forward example in §5-6-1); [alternatively: **RM**, §1.2, 1.3 and 1.4, pp 4-12].

(d) Implicit schemes and their stability advantage (implicit scheme for advection equation **HW**, §5-6-3 p 132 and first half of p 133; [alternatively, implicit scheme for diffusion equation **RM**, §1.5, pp 16-18];

(e) Stiff problems (wikipedia, or a local copy, highlighted definitions and examples of eqns 1-3, 10-15).

(f) Preserving conservation laws of the continuous equations in the finite difference approximation, examples: advection and diffusion, advantage of writing the finite difference equations in flux form, and demonstrating staggered grids using the same example (notes).

(g) Consistent formulation of finite-difference boundary conditions: prescribed flux/ value for diffusion equation, fixed vs stress-free end of a string for wave equation, location of end points on grid as function of type of boundary condition (notes).

(h) Finite element methods (Burden and Faires, Numerical Analysis, 8th edition, §12.4 pp 721-733).

(i) Artificial numerical dispersion and dissipation (Zhiming's notes from apm111). Demonstrate using `numerical_dispersion_and_dissipation.m`. [Alternatively, Durran, Numerical methods for wave equations in geophysical fluid dynamics, §2.4.1, 2.4.2, 2.4.3]

(j) Spectral methods (Durran, Numerical methods for wave equations in geophysical fluid dynamics, §4.2 p 176-184 or so).