# PRESERVATION OF CONDITIONALLY PERIODIC MOVEMENTS WITH SMALL CHANGE IN THE HAMILTON FUNCTION*

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## ABSTRACT

This paper is a translation of Kolmogorov's original article announcing the theorem now known as the KAM theorem.

## THEOREM AND DISCUSSION OF PROOF

Let us consider in the 2s-dimensional phase space of a dynamic system with s degrees of freedom the region G, represented as the product of an s-dimensional torus, T, and a region S, of a Euclidian s-dimensional space. We will designate the points of the torus, T, by the circular coordinates $q_1, \ldots, q_s$ (replacing $q_\alpha$ with $q'_\alpha = q_\alpha + 2\pi$ does not change points q), and the coordinates of the points, p, of S we will designate as $p_1, \ldots, p_s$. We will assume that in region G, in the coordinates $(q_1, \ldots, q_s, p_1, \ldots, p_s)$ the equations of motion have the canonical form

$$\frac{dq_\alpha}{dt} = \frac{\partial}{\partial p_\alpha} H(q,p), \quad \frac{dp_\alpha}{dt} = -\frac{\partial}{\partial q_\alpha} H(q,p). \tag{1}$$

The Hamilton function, H, is further assumed as dependent on the parameter $\theta$ and determined for all $(q,p) \in G$, $\theta \in (-c; +c)$, but not time-dependent. Moreover, further considerations require fairly significant restrictions on the smoothness of the function $H(q, p, \theta)$, stronger than infinite differentiability. For simplicity, in the following it is assumed that the function $H(q, p, \theta)$ is analytic over the set of variables $(q, p, \theta)$.

Summation over the Greek indices is further assumed to be from 1 to s. The usual vector designations $(x,y) = \sum_\alpha x_\alpha y_\alpha$, $|x| = +\sqrt{(x,x)}$ are used. A whole number vector indicates a vector for which all the components are whole numbers. The set of points (q, p) of G with

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$p = c$ are designated by $T_c$. In theorem 1 it is assumed that S contains the point $p = 0$, i.e., $T_0 \subseteq G$.

Theorem 1. Let

$$H(q,p,0) = m + \sum_{\alpha} \lambda_\alpha p_\alpha + \frac{1}{2} \sum_{\alpha\beta} \Phi_{\alpha\beta}(q) p_\alpha p_\beta + O(|p|^3), \qquad (2)$$

where m and $\lambda_\alpha$ are constants and for a certain choice of constants $c > 0$ and $\eta > 0$ for all whole-number vectors, n, the inequality

$$(n,\lambda) \geq \frac{c}{|n|^\eta} \quad . \qquad (3)$$

is satisfied.

Let, moreover, the determinant composed of the average values

$$\varphi_{\alpha\beta}(0) = \frac{1}{(2\pi)^s} \int_0^{2\pi} \cdots \int_0^{2\pi} \Phi_{\alpha\beta}(q)\, dq_1 \ldots dq_s$$

of the functions

$$\Phi_{\alpha\beta}(q) = \frac{\partial^2}{\partial p_\alpha \partial p_\beta} H(q,0,0)$$

be different from zero:

$$|\varphi_{\alpha\beta}(0)| \neq 0. \qquad (4)$$

Then there exist analytical functions $F_\alpha(Q, P, \theta)$ and $G\ (Q, P, \theta)$ which are determined for all sufficiently small $\theta$ and for all points $(Q, P)$ of some neighborhood, V, of the set $T_0$, which bring about a contact transformation

$$q_\alpha = Q_\alpha + \theta F_\alpha(Q,P,\theta), \quad p_\alpha = P_\alpha + \theta G\ (Q,P,\theta)$$

of V into $V' \subseteq G$, which reduced H to the form

$$H = M(\theta) + \sum_{\alpha} \lambda_\alpha P_\alpha + O(|P|^2) \qquad (5)$$

($M(\theta)$ does not depend on Q and P).

It is easy to grasp the meaning of Theorem 1 for mechanics. It indicates that an s-parametric family of conditionally periodic motions

$$q_\alpha = \lambda_\alpha t + q_\alpha^{(0)}, \quad p_\alpha = 0,$$

which exists at $\theta = 0$ cannot, under conditions (3) and (4), disappear as a result of a small change in the Hamilton function $H$: there occurs only a displacement of the s-dimensional torus, $T_0$, around which the trajectories of these motions run, into the torus $P = 0$, which remains filled by the trajectories of conditionally periodic motions with the same frequencies $\lambda_1, \ldots, \lambda_s$.

The transformation

$$(Q, P) = K_\theta(q, p),$$

the existence of which is confirmed in Theorem 1, can be constructed in the form of the limit of the transformations

$$(Q^{(k)}, P^{(k)}) = K_\theta^{(k)}(q, p),$$

where the transformations

$$(Q^{(1)}, P^{(1)}) = L_\theta^{(1)}(q, p), \quad (Q^{(k+1)}, P^{(k+1)}) = L_\theta^{(k+1)}(Q^{(k)}, P^{(k)})$$

are found by the "generalized Newton method" (see Ref. 1). In this note we confine ourselves to the construction of the transformation: $K_\theta^{(1)} = L_\theta^{(1)}$, which itself permits grasping the role of conditions (3) and (4) of Theorem 1. Let us apply the transformation $L_\theta^{(1)}$ to the equations

$$Q_\alpha^{(1)} = q_\alpha + \theta Y_\alpha(q),$$

$$p_\alpha = P_\alpha^{(1)} = \theta\left\{\sum_\beta P_\beta^{(1)}\frac{\partial Y_\beta}{\partial q_\alpha} + \xi_\alpha + \frac{\partial}{\partial q_\alpha}X(q)\right\} \qquad (6)$$

(it is easy to verify that this is a contact transformation) and seek the constants $\xi_\alpha$ and $\zeta$ and the functions $X(q)$ and $Y_\beta(q)$, starting from the requirement that

$$H = m + \sum_\alpha \lambda_\alpha p_\alpha + \frac{1}{2}\sum_{\alpha\beta}\Phi_{\alpha\beta}(q)p_\alpha p_\beta +$$
$$+ \theta\left\{A(q) + \sum_\alpha B_\alpha(q)p_\alpha\right\} + O(|p|^3 + \theta|p|^2 + \theta^2) \qquad (7)$$

take the form

$$H = m + \theta\zeta + \sum \lambda_\alpha P_\alpha^{(1)} + O(|P^{(1)}|^2 + \theta^2). \qquad (8)$$

Substituting (6) into (7), we get

$$H = m + \sum_\alpha \lambda_\alpha P_\alpha^{(1)} + \theta \left\{ A + \sum_\alpha \lambda_\alpha \left( \xi_\alpha + \frac{\partial X}{\partial q_\alpha} \right) \right\} +$$

$$+ \theta \sum_\alpha P_\alpha^{(1)} \left\{ B_\alpha + \sum_\beta \phi_{\alpha\beta}(q) \left( \xi_\beta + \frac{\partial X}{\partial q_\beta} \right) + \sum_\beta \lambda_\beta \frac{\partial Y_\beta}{\partial q_\beta} \right\} + O(|P^{(1)}|^2 + \theta^2).$$

Thus, our requirement (8) reduces to the equations

$$A + \sum_\alpha \lambda_\alpha \left( \xi_\alpha + \frac{\partial X}{\partial q_\alpha} \right) = \zeta \tag{9}$$

$$B_\alpha + \sum_\beta \phi_{\alpha\beta} \left( \xi_\beta + \frac{\partial X}{\partial q_\beta} \right) + \sum_\beta \lambda_\beta \frac{\partial Y_\alpha}{\partial q_\beta} = 0. \tag{10}$$

being fulfilled.

Let us introduce the functions

$$Z_\alpha(q) = \sum_\beta \phi_{\alpha\beta}(q) \frac{\partial}{\partial q_\beta} X(q). \tag{11}$$

Expanding the functions $\phi_{\alpha\beta}$, $A$, $B_\alpha$, $X$, $Y_\alpha$, $Z_\alpha$ into a Fourier series of the type

$$X(q) = \sum x(n) e^{i(n,q)}$$

and assuming for definiteness that

$$x(0) = 0, \quad y(0) = 0, \tag{12}$$

we get for the remaining Fourier coefficients $x(n)$, $y_\alpha(n)$, and $z_\alpha(n)$ and constants $\xi_\alpha$ and $\zeta$ of the equation which are relevant to the determination

$$a(0) + \sum \lambda_\alpha \xi_\alpha = \zeta, \tag{13}$$

$$a(n) + (n,\lambda)\, x(n) = 0 \quad \text{for } n \neq 0, \tag{14}$$

$$b_\alpha(0) + \sum_\beta \varphi_\alpha(0) \xi_\beta + z_\alpha(0) = 0, \tag{15}$$

$$b_\alpha(n) + \sum_\beta \varphi_{\alpha\beta}(n) \xi_\beta + z_\alpha(n) + (n,\lambda)\, y_\alpha(n) = 0 \quad \text{for } n \neq 0. \tag{16}$$

It is easy to see that the system (11) - (16) is unambiguously solved under conditions (3) and (4). Condition (3) is important in the determination of $x(n)$ from (14), and in the determination of $y_\alpha(n)$ from (16). Condition (4) is important in the determination of $\xi_\beta$ from (15). Since, as $|n|$ increases, the coefficients of the Fourier series of the analytical functions $\varphi_{\alpha\beta}$, A, and $B_\alpha$ have an order of decrease not less than $\rho^{|h|}$, $\rho < 1$, then from condition (3) there results not only the formal solvability of equations (13) - (16) but also the convergence of the Fourier series for the functions X, $Y_\alpha$, and $Z_\alpha$ and the analyticity of these functions. The construction of further approximations is not associated with new difficulties. Only the use of condition (3) for proving the convergence of the recursions, $K_\beta^{(k)}$, to the analytical limit for the recursion $K_\beta$ is somewhat more subtle.

The condition of the absence of "small denominators" (3) should be considered, "generally speaking," as fulfilled, since for any $\eta > s - 1$ for all points of an s-dimensional space $\lambda = (\lambda_1, \ldots, \lambda_s)$ except the set of Lebesque measure zero it is possible to find $c(\lambda)$, for which

$$(n, \lambda) \geq \frac{c(\lambda)}{|n|^\eta}$$

whatever the integers $n_1, n_2, \ldots, n_s$ were[2]. It is also natural to consider condition (4) as, "generally speaking," fulfilled. Since

$$\varphi_{\alpha\beta}(0) = \frac{\partial}{\partial p_\alpha} \lambda_\beta(0),$$

where

$$\lambda_\beta(p) = \frac{1}{(2\pi)^s} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{dq_\beta}{dt} \, dq_1 \ldots dq_s$$

is the frequency averaged over the coordinate $q_\beta$ with fixed momenta $p_1, \ldots, p_s$, condition (3) means that the Jacobian of the average frequencies over the momenta is different from zero.

Let us turn now to a consideration of the special case where $H(q, p, 0)$ depends only on p, i.e., $H(q, p, 0) = W(p)$. In this case, for $\theta = 0$ each torus, $T_p$, consists of the complete trajectories of the conditionally periodic movements with frequencies

$$\lambda_\alpha(p) = \frac{\partial W}{\partial p_\alpha}.$$

If the Jacobian

$$J = \left|\frac{\partial \lambda_\alpha}{\partial p_\beta}\right| = \left|\frac{\partial^2 W}{\partial p_\alpha \partial p_\beta}\right| \tag{17}$$

is different from zero, then it is possible to apply Theorem 1 to almost all tori, $T_p$. There arises the natural hypothesis that at small $\theta$ the "displaced tori" obtained in accordance with Theorem 1 fill a larger part of region G. This is also confirmed by Theorem 2, pointed out later. In the formulation of this theorem we will consider the region S to be bounded and will introduce into the consideration the set, $M_\theta$, of those points $(q^{(0)}, p^{(0)}) \varepsilon G$ for which the solution

$$q_\alpha(t) = f_\alpha(t; q^{(0)}, p^{(0)}, \theta), \quad p_\alpha(t) = G_\alpha(t; q^{(0)}, p^{(0)}, \theta)$$

of the system of equations (1) with initial conditions

$$q_\alpha(0) = q_\alpha^{(0)}, \quad p_\alpha(0) = p_\alpha^{(0)}$$

leads to trajectories not moving out of region G with change in t from $-\infty$ to $+\infty$, and conditionally periodic with periods $\lambda_\alpha = \lambda_\alpha(q^{(0)}, p^{(0)}, \theta)$, i.e., it has the form

$$f_\alpha(t) = \varphi_\alpha(e^{i\lambda_1 t}, \ldots, e^{i\lambda_s t}), \quad g_\alpha(t) = \psi_\alpha(e^{i\lambda_1 t}, \ldots, e^{i\lambda_s t}).$$

Theorem 2. If $H(q, p, 0) = W(p)$ and determinant (17) is not equal to zero in region S, then for $\theta \to 0$ the Lebesque degree of the set $M_\theta$ converges to the complete degree of region S.

Apparently, in the usual sense of the phrase, "general case" is when the set $M_\theta$ at all positive $\theta$ is everywhere dense. In such a case the complications arising in the theory of analytical dynamic systems are indicated more specifically in my note.[3]

## REFERENCES

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