APM 203 Homework #6 Solutions

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Problem #1

Calculate a numerical approximation for

$$\lim_{n \to \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n}$$

(a) for the map $x_{n+1} = r \sin(\pi x_n)$ $0 \le r \le 1$,

|| Note that for $r > 1$, map may diffuse out of basin of attraction $(0,1)$. Thus, restriction.

|| Also, around $x = \frac{1}{2}$, $x_{n+1} \approx r - r \cdot \frac{\pi^2}{2} x_n^2$ i.e. the map is unimodular. Thus, we expect the limit to converge to the Feigenbaum constant.

Estimates found using bif_sin.m program.
(see code on attached page)

$r_1 = 1/\pi \approx 0.3183\ldots$ (exact)

$r_2 \approx 0.7187\ldots^{*}$

$r_3 \approx 0.8328\ldots$

$r_4 \approx 0.8585\ldots$

$r_5 \approx 0.8641\ldots$

$r_6 \approx 0.8653\ldots$

δ estimates:

$(r_2 - r_1)/(r_3 - r_2) = 3.507\ldots$

$(r_3 - r_2)/(r_4 - r_3) = 4.4503\ldots$

$(r_4 - r_3)/(r_5 - r_4) = 4.5804\ldots$

$\boxed{(r_5 - r_4)/(r_6 - r_5) = 4.6667\ldots}$

... getting better!!

<u>Note</u>: this can be obtained another way, be solving $-1 = \pi r_2 \cos\sqrt{\pi^2 r_2^2 - 1}$.

(b) for the map $x_{n+1} = r - x_n^4$. We may see a few things analytically:

‖ 1-cycle bifurcation occurs at $r_1 = \frac{1}{4^{4/3}} - \frac{1}{4^{1/3}}$

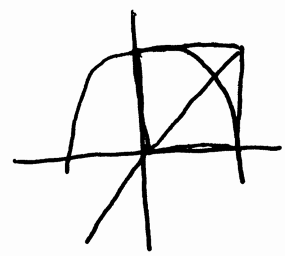
$\simeq -0.4725...$

and becomes superstable at $r = 0$.

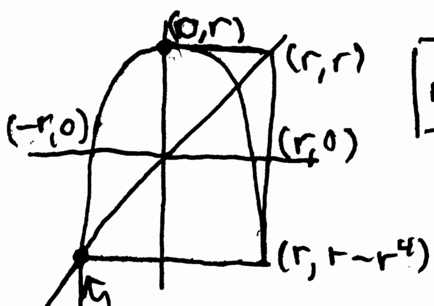
‖ 2-cycle bifurcation occurs when $\frac{d}{dx}(r - x^4) = -1$ and $x = r - x^4$. using this $\Rightarrow$ $r = \frac{1}{4^{4/3}} + \frac{1}{4^{1/3}}$

$\simeq 0.7875...$

‖ 2-cycle becomes <u>superstable</u> at $r = 1$



‖ At $r = 2^{1/3}$, orbits begin to diffuse to $-\infty$ as $n \to \infty$.



$\boxed{r = 2^{1/3}}$

f.p. $\Rightarrow -r = r - r^4$ or $\underline{\underline{r = 2^{1/3}}}$.

<u>Experimental bifurcation points:</u>

$r_1 = -0.4725...$ (exact)

$r_2 = 0.7875...$ (exact)

$r_3 = 1.1196...$

$r_4 = 1.1617...$

$r_5 = 1.1675...$

$r_6 = 1.1683...$

thus $\delta_1 = (r_2 - r_1)/(r_3 - r_2) = 3.79...$

$\delta_2 = (r_3 - r_2)/(r_4 - r_3) = 7.888...$

$\delta_3 = (r_4 - r_3)/(r_5 - r_4) = 7.259...$

$\delta_4 = (r_5 - r_4)/(r_6 - r_5) = \underline{\underline{7.250...}}$

Not Feigenbaum #! This is a different maximum!

Problem #2

(a) show that if $g(x)$ is a f.p. of T,

i.e. $g(x) = -\alpha g\left[g\left(\frac{x}{-\alpha}\right)\right] \equiv T[g]$ $(*)$

so is $\mu g\left[\frac{x}{\mu}\right]$.

<u>Sol'n:</u> $T\left[\mu g\left(\frac{x}{\mu}\right)\right] = -\alpha\mu g\left[\frac{1}{\mu} \cdot \mu g\left(\frac{x}{-\mu\alpha}\right)\right]$

$= -\alpha\mu g\left[g\left(\frac{x}{-\alpha\mu}\right)\right]$

using $(*)$ $\Rightarrow$ this is $\mu g\left(\frac{x}{\mu}\right)$ ∎.

(b) $g(x)$ crosses $y = \pm x$ infinite # of times:

if x^* is a f.p., $g(x^*) = x^*$, then

$g(-\alpha x^*) = -\alpha g\left[g(x^*)\right] = -\alpha g(x^*)$

$= -\alpha x^*$

but $g(-\alpha x^*) = g(\alpha x^*)$. Thus

$g(z) = \pm z$ at all $\pm\alpha^n x^*$.

(c) let $g(0) = 1$. $g = T[g] \Rightarrow 1 + cx^2 = -\alpha\left(1 + c\left(1 + c\left(\frac{x}{\alpha}\right)^2\right)^2\right)$.

or $1 + cx^2 = -\alpha\left(1 + c\left(1 + 2c\frac{x^2}{\alpha^2} + O(x^4)\right)\right)$

thus, $1 = -\alpha(1+c)$ and $c = -\alpha \cdot \frac{2c^2}{\alpha^2}$ $\Rightarrow$ $c = -1.618\ldots$, $\alpha = -3.24\ldots$ <u>Not bad</u>.

Problem #3

Following Schuster (page 46) begin with the definition of the function

$$g_i(x) \equiv \lim_{n\to\infty} (-\alpha)^n f^{2^n}_{R_{n+i}}\left[\frac{x}{(-\alpha)^n}\right] \qquad i=0,1,\ldots$$

which are functions with 2^i- superstable cycles. How are they related?

$$g_{i-1}(x) = \lim_{n\to\infty} (-\alpha)^n f^{2^n}_{R_{n+i-1}}\left[\frac{x}{(-\alpha)^n}\right]$$

$$= \lim_{n\to\infty} (-\alpha)(-\alpha)^{n-1} f^{2^{n-1+1}}_{R_{n-1+i}}\left[-\frac{1}{\alpha}\cdot\frac{x}{(-\alpha)^{n-1}}\right]$$

$$= \lim_{m\to\infty} (-\alpha)(-\alpha)^m f^{2^m}_{R_{m+i}}\left[f^{2m}_{R_{m+i}}\left[\frac{1}{-\alpha}\cdot\frac{x}{(-\alpha)^m}\right]\right]$$

where we've recast $n-1\to m$. and we've used $f^{2^m}\circ f^{2^m} = f^{2^{m+1}}$.

thus, $$g_{i-1}(x) = \lim_{m\to\infty} (-\alpha)(-\alpha)^m f^{2^m}_{R_{m+i}}\left\{\underbrace{\frac{1}{(-\alpha)^m}(-\alpha)^m f^{2^m}_{R_{m+i}}\left[-\frac{1}{\alpha}\cdot\frac{x}{(-\alpha)^m}\right]}_{g_i(x) \text{ in the limit } m\to\infty.}\right\}$$

$$= \lim_{m\to\infty} (-\alpha)(-\alpha)^m f^{2^m}_{R_{m+i}}\left[\frac{g_i(x)}{(-\alpha)^m}\right]$$

$$= -\alpha\cdot g_i\left[g_i\left(-\frac{x}{\alpha}\right)\right]$$ ∎

Problem #4

(a) Everyone understood this question — I'll do some of part b:

(b) I'll calculate α approximately using the period-doubling transformation equation which arose from renormalization analysis for general maps with an arbitrary maximum:

$$g(x) = \alpha\, g^2\left(\frac{x}{\alpha}\right).$$

But this time around, we'll assume $g(x) \simeq 1 + c_4 x^4$ with $c_2 = 0$ (if f is quartic, it follows g must be quartic to lowest order.)

thus,
$$1 + c_4 x^4 = \alpha\left[1 + c_4\left(1 + c_4\left(\frac{x}{\alpha}\right)^4\right)^4\right]$$
$$= \alpha(1 + c_4) + \alpha\, c_4 \cdot 4 c_4 \frac{x^4}{\alpha^4} + \mathcal{O}(x^8)$$

$$\Rightarrow \quad 1 = \alpha(1 + c_4)\,; \qquad c_4 = 4\alpha\, c_4^2/\alpha^4 = 4c_4^2/\alpha^3$$

thus, $\alpha^3 = 4c_4$ or $1 = \alpha\left(1 + \frac{\alpha^3}{4}\right)$

$\Rightarrow \alpha = -1.835\ldots$ and $c_4 = -1.545\ldots$