Applied Math 203 Homework #6 Solution

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(1) $\dot{x} = \sigma(y - x)$

$\dot{y} = rx - y - xz$

$\dot{z} = xy - bz$

fixed points: $\vec{x}^* = \vec{0}$, $\vec{x}^* = (\pm\varsigma, \pm\varsigma, r-1)$, $\varsigma \equiv \sqrt{(r-1)b}$

At $1 < r < r_H$, the fixed points $C_\pm$ are stable spirals; they become unstable at $r > r_H$ and the eigenvalues cross the imaginary axis. At $r = r_H$, the eigenvalues of the Jacobian at $C_\pm$ are pure imaginary.

$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ (r-z) & -1 & -x \\ y & x & -b \end{pmatrix}$. At $C_\pm$, $J = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \mp\varsigma \\ \pm\varsigma & \pm\varsigma & -b \end{pmatrix}$

The characteristic eqn is indep of $\pm$ (both f.p.s become unstable at same r_H):

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma) b\lambda + 2b\sigma(r-1) = 0 \qquad (1)$$

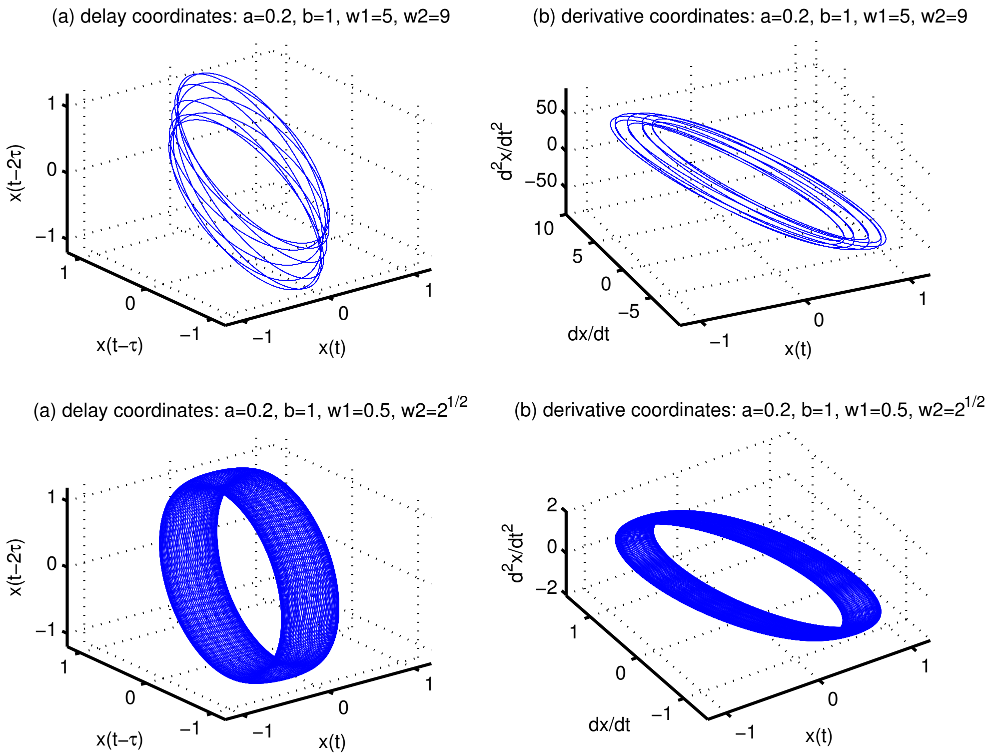
Let $\lambda = i\omega$, with ω real, and real and imag parts of (1) give 2 eqns. Eliminate ω between them to get

$$\boxed{r_H = \sigma \frac{(\sigma + b + 3)}{(\sigma - b - 1)}}$$

2.

$x(t) = a\sin(\omega_1 t) + b\sin(\omega_2 t)$

The minimum values of n, m that satisfy $n\omega_1 = m\omega_2$ tell how many times the trajectory has to wrap around before repeating itself. I used $b > a$ and $\frac{\omega_1}{\omega_2} = \frac{5}{9}$, so the trajectory had to wrap 5 times around the wide axis of the torus while wrapping 9 times around the narrow axis before repeating itself. When $\frac{\omega_1}{\omega_2}$ is irrational, the trajectory is quasiperiodic and never repeats itself. It can be readily demonstrated, however, that this system does not have the exponential sensitivity to initial conditions required for it to be chaotic.

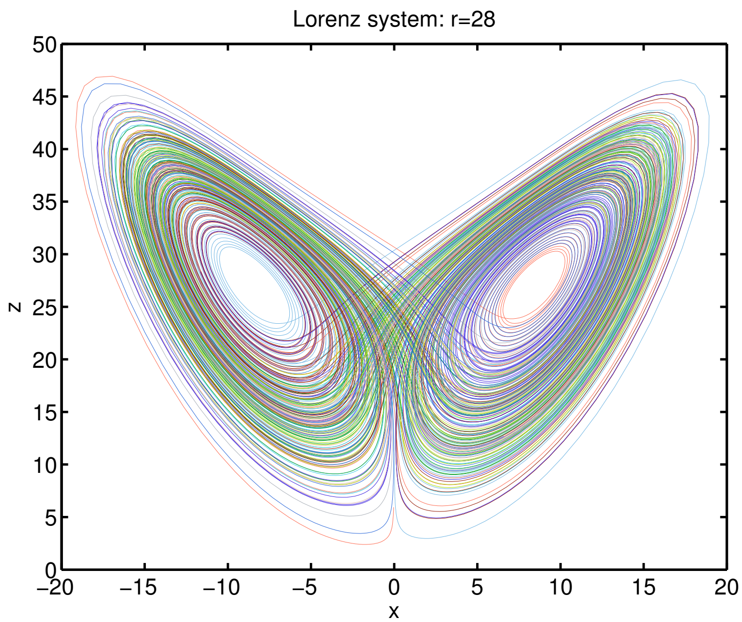


3.

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= rx-y-xz\\ \dot{z} &= xy-bz\end{aligned}$$

Using $\sigma = 10$, $b = 8/3$, $r > 0$.

(a) The Lorentz attractor is plotted below for $r = 28$. I used 10 different initial conditions and plotted them each in a different color for times 50-100. Note that colors don't remain adjacent.

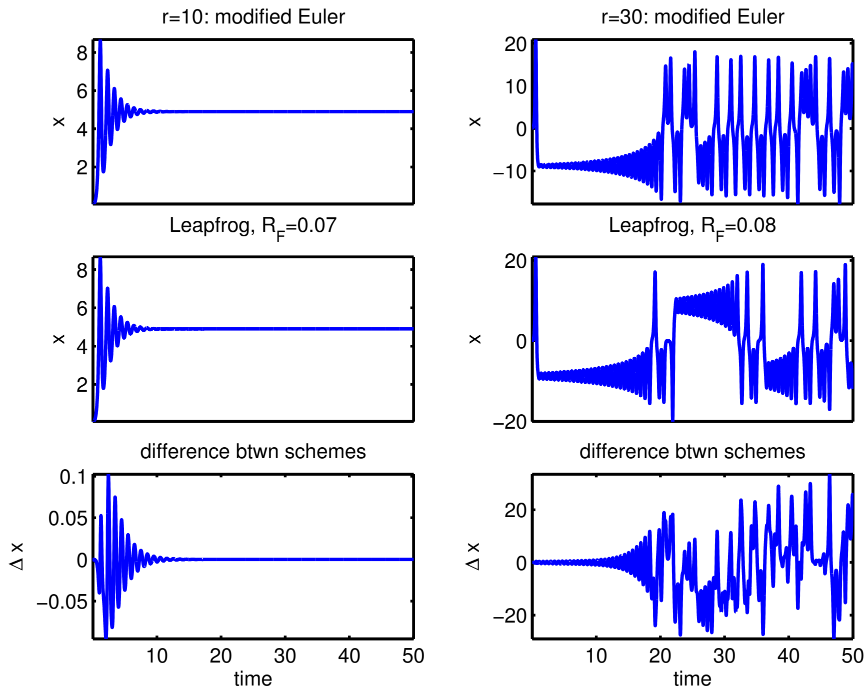


(b) The leapfrog scheme is numerically unstable. A Robert filter smooths the $n-1$ step before the $n+1$ step jumps from it. The first few values must be evaluated by using the forward Euler scheme for a few steps after the initial condition. Then the leapfrog scheme is applied as follows:

$$X_{n+1} = X_{n-1} + F(C_n)2\Delta t + R_F(X_n - 2X_{n-1} + X_{n-2})$$

When $R_F > 0$, a Robert filter is being applied. One typically chooses the smallest value of R_F that leads to a non-exploding solution. For this system it's $R_F \approx 0.1$ (depends on r).

When the system is not chaotic, the two numerical schemes give similar results. When the system is chaotic, however, trajectories are very sensitive to initial conditions, so slight differences in the integration of each step will lead to large differences in the trajectories. Here the two different methods of numerical integration lead to trajectories that diverge after about $t = 15$.



(c) With $r = 1e - 5$ and $X_0 = (0.1, 0, 0)$, the system is very near a very stable fixed point. The trajectory integrated with the leapfrog scheme and no Robert blows up quickly after $t \approx 1.5$.

The location of the Hopf bifurcation ranges between about 26 and 39 as the Robert filter is varied between 0.1 and 0.5, with r_H increasing monotonically as R_F is increased (note that the theoretical value with these parameters is $r_H = 24.7$). This implies that although numerical integration is often useful, it may not be a good way to accurately find critical points for chaotic systems. Here apparently the stabilizing effect of the Robert filter delays the onset of chaos.