APPLIED MATH 203 HOMEWORK #9 SOLUTIONS

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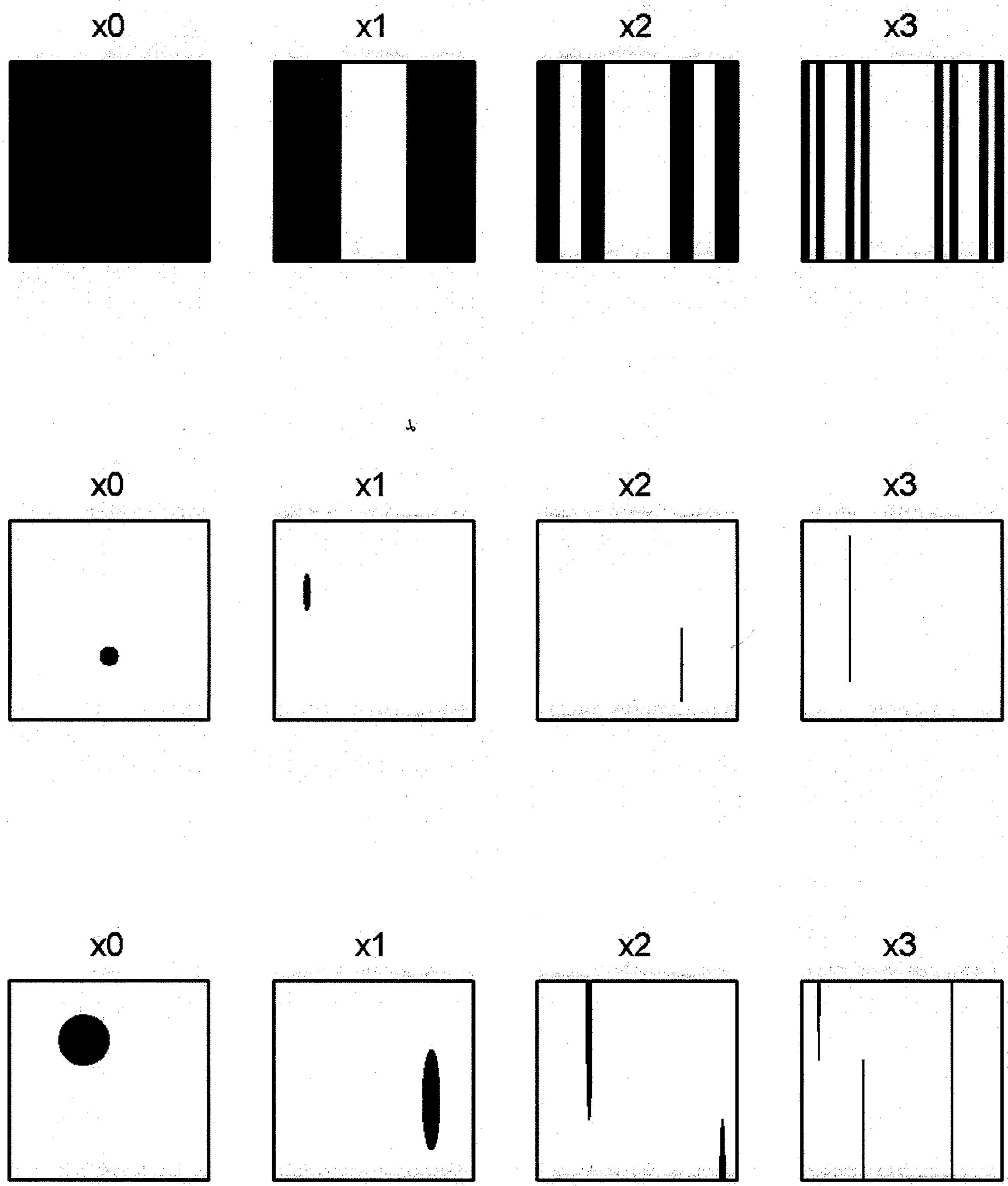
① Baker map: $B(x,y) = \begin{cases} (\frac{1}{3}x, 2y) & 0 \le y \le \frac{1}{2} \\ (\frac{1}{3}x + \frac{2}{3}, 2y-1) & \frac{1}{2} < y \le 1 \end{cases}$

a) On following page, the first three iterations mapping the unit square are plotted. Note that the pattern becomes a Cantor set (middle third) in x as we continue iterating ad infinitum. The first three iterations for two different circles of initial conditions are also plotted. When the circle is mapped onto the $y = \frac{1}{2}$ line, it will be mapped to the edges in the next iteration. Any points mapped to $y=0$ remain at $y=0$.

b) Each step scales x by $\frac{1}{3}$ and y by 2, so a patch of initial conditions $(\vec{x}_0, \vec{y}_0)$ is scaled by $((\frac{1}{3})^n, 2^n) = (e^{n \ln(1/3)}, e^{n \ln 2})$ after n steps. Hence the Lyapunov exponents are $\boxed{\lambda_1 = \ln 2; \ \lambda_2 = -\ln 3}$. This exponential behavior only applies to patches that don't hit the $y = \frac{1}{2}$ line. Circles are stretched in $\hat{y}$ and compressed in $\hat{x}$ into ever narrower and longer ellipses.

1a: Bakers map



(2) $K_n = -\sum_{i_0,\dots,i_n} P_{i_0,\dots,i_1} \log P_{i_0,\dots,i_1}$

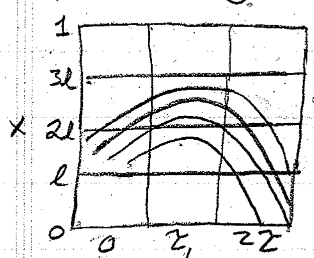
$K = \lim_{\tau\to 0} \lim_{l \to 0} \lim_{N\to\infty} \frac{1}{N\tau} \sum_{n=0}^{N-1} (K_{n+1} - K_n)$

Consider a 1D dynamic system. Make a grid in space and time, and every trajectory can be described by the series of grid boxes it passes through. We start with a spread of initial conditions filling the 1D spatial domain. The fraction of trajectories passing through the series of boxes

i_0 (at $t=0$) $\to$ i_1 (at $t=\tau$) $\to$ i_2 (at $t=2\tau$)

is defined as $P_{i_0 i_1 i_2}$. Since initially the trajectories are evenly spread, $P_{i_0} = l$ for all initial boxes i_0.

a) Periodic system (or any system where initially adjacent trajectories remain adjacent).



Most paths (i.e., series of boxes) don't occur for any trajectory and hence have $P=0$, so for these $P \log P = \lim_{x\to 0} x \log x = 0$. In the limit $l \to 0$, there is only one series of boxes that all trajectories starting in a box i_0 will pass through.

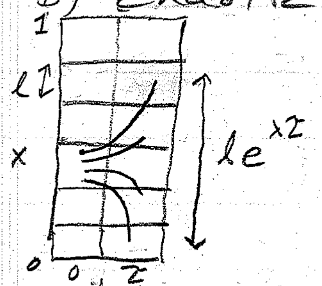
$P_{i_0} = l$, and here $P_{i_0 i_1} = \begin{cases} l & \text{one path} \\ 0 & \text{other paths} \end{cases}$

$K_0 = -\sum_{i_0} P_{i_0} \log P_{i_0} = -\left(\frac{1}{l}\right) l \log l$ [there are $\frac{1}{l}$ initial boxes]

$K_1 = -\sum_{i_0 i_1} P_{i_0 i_1} \log P_{i_0 i_1} = -\left(\frac{1}{l}\right) l \log l = \boxed{-\log l = K_n} \ \forall n$

So $K_{n+1} - K_n = 0 \ \forall n$, and $\boxed{K=0}$. Note that this readily generalizes to a n-dimensional system.

② b) chaotic system (initial error grows)



The trajectories that start in the box i_o spread to evenly fill $le^{\lambda\tau}$ boxes at time τ. In 1D,

$P_{i_0} = l$. Given i_0,

$$P_{i_1} = \begin{cases} e^{-\lambda\tau} & e^{\lambda\tau} \text{ boxes} \\ 0 & \text{other boxes} \end{cases}$$

So $P_{i_0 i_1} = \begin{cases} le^{-\lambda\tau} \\ 0 \end{cases}$, and

$$K_0 = -\sum_{i_0} P_{i_0} \log P_{i_0} = -\log l$$

$$K_1 = -\sum_{i_0 i_1} P_{i_0 i_1} \log P_{i_0 i_1} = -\underbrace{\left(\tfrac{1}{l}e^{\lambda\tau}\right)}_{\#\text{ of terms in sum}} le^{-\lambda\tau} \log le^{-\lambda\tau} = \lambda\tau - \log l$$

After more steps, the trajectories keep spreading, so $P_{i_0,\dots,i_n} = le^{-n\lambda\tau}$ and $\boxed{K_n = n\lambda\tau - \log l}$

What about in 3D? Note that if $\lambda<0$, there is only one possible path from a given i_0 (Kolmogorov entropy ignores that fact that trajectories filling i_0 only fill part of i_1), so $K=0$ identically to the periodic system. With $\lambda_1>0$, $\lambda_2>0$, $\lambda_3<0$,

$P_{i_0} = l^3$

$$P_{i_0 i_1} = \begin{cases} e^{-(\lambda_1+\lambda_2)\tau} & e^{(\lambda_1+\lambda_2)\tau} \text{ boxes} \\ 0 & \text{other boxes} \end{cases}$$

Following steps as above, $\boxed{K_n = n(\lambda_1+\lambda_2)\tau - 3\log l}$

$$\Delta K_n \equiv K_{n+1} - K_n = (\lambda_1+\lambda_2)\tau$$

$$K = \lim_{\tau\to 0}\lim_{l\to 0}\lim_{N\to\infty} \frac{1}{N\tau}\sum_{n=0}^{N-1}\Delta K = \lim_{\tau,l,N} \frac{1}{N\tau} \underbrace{N}_{\#\text{ of terms in sum}} (\lambda_1+\lambda_2)\tau$$

$K = \lim_{\tau,l,N} (\lambda_1+\lambda_2)$ [all limits trivial]

$$\boxed{K = \lambda_1+\lambda_2}$$

② c) <u>Stochastic system</u> (every point in trajectory is random and independent of previous path).

In 2D, $P_{i_0} = \ell$. Given P_{i_0}, $P_{i_1} = \ell$, so $P_{i_0 i_1} = \ell^2$, and $P_{i_0, \ldots, i_n} = \ell^n$

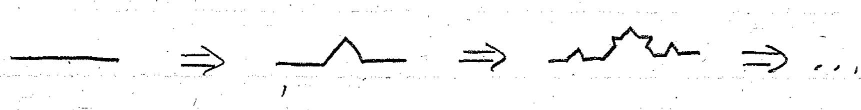
$$K_n = -\sum_{i_0, \ldots, i_n} P \log P = -\underbrace{(\ell^{-n})}_{\#\text{ of terms}} \ell^n \log \ell^n = -n \log \ell$$

$$\Delta K = -\log \ell$$

$$K = \lim_{\tau, \ell, N} \frac{1}{N\tau} \sum_n \Delta K = \lim_{\tau, \ell, N} \frac{1}{N\tau} [N(-\log \ell)]$$

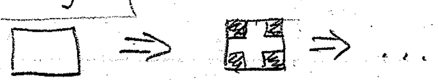
$$K = \lim_{\tau \to 0} \lim_{\ell \to 0} \left(-\frac{\log \ell}{\tau}\right) \to \infty$$

$$\boxed{K = \infty}$$

③ a) i) 

The Koch curve is composed of $m=4$ copies of itself, each scaled down by $r=3$.

$$\boxed{d_{sim} = \frac{\log 4}{\log 3}}$$

ii) 

This object is made of $m=4$ copies of itself, each scaled down by $r=3$

$$\boxed{d_{sim} = \frac{\log 4}{\log 3}}$$

Note that the box counting dimension gives the same answer for these two fractals (as it does for most fractals).

b) See OH.

③ c) A line is $0 \le x \le 1$, so there are 2 edges, $x=0$ and $x=1$. A square is $0 \le x \le 1$, $y \le 0 \le 1$, so there are 4 edges, $x = 0,1$ and $y = 0,1$. So an n-dimensional hypercube has $2n$ edges (or faces). We're removing a piece from each edge and the center, or $(2n+1)$ pieces. This hypercube is composed of 3^n smaller hypercubes scaled down by $r=3$, and we've removed $2n+1$ hypercubes leaving $m = 3^n - (2n+1)$ remaining.

$$\boxed{d_{sim} = \frac{\log(3^n - 2n - 1)}{\log 3}}$$

Note that when $n \to \infty$, $d_{sim} \to n$, so the set approaches the dimension of the full hypercube.