APPLIED MATH 203 Homework 10 SOLUTIONS

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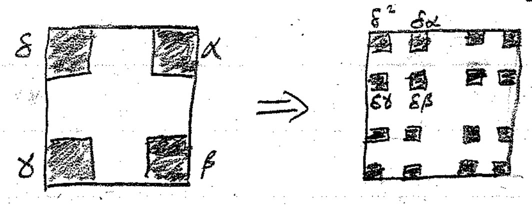
① 

$\alpha+\beta+\gamma+\delta=1$

$\alpha,\beta,\gamma,\delta>0$

a) Let $\varepsilon = \frac{1}{3^n}$

$$I(q,\varepsilon) = \sum_{i=1}^{N(\varepsilon)} \mu_i^q = (\alpha^q+\beta^q+\gamma^q+\delta^q)^n$$

$$D_q = -\frac{1}{q-1}\lim_{\varepsilon\to 0}\frac{\log I}{\log 1/\varepsilon} = -\frac{1}{1-q}\lim_{n\to\infty}\frac{\log(\alpha^q+\beta^q+\gamma^q+\delta^q)^n}{\log 3^n}$$

$$\boxed{D_q = \frac{1}{1-q}\,\frac{\log(\alpha^q+\beta^q+\gamma^q+\delta^q)}{\log 3}}$$

$D_0 = \frac{\log 4}{\log 3}$ = box-counting dimension

$D_1 = \frac{\alpha\log\alpha+\beta\log\beta+\gamma\log\gamma+\delta\log\delta}{\log(1/3)}$ = information dimension [by L'Hôpital]

Also, $D_{-\infty} = \lim_{q\to-\infty} D_q = -\frac{\log\max(\alpha,\beta,\gamma,\delta)}{\log 3}$

$D_{\infty} = \lim_{q\to\infty} D_q = -\frac{\log\min(\alpha,\beta,\gamma,\delta)}{\log 3}$

See plot on next page.

b) $f(\alpha(q)) = -(q-1)D_q + q\,\alpha(q)$ (1)

$\alpha(q) = \frac{d}{dq}\left[(q-1)D_q\right]$

In this problem,

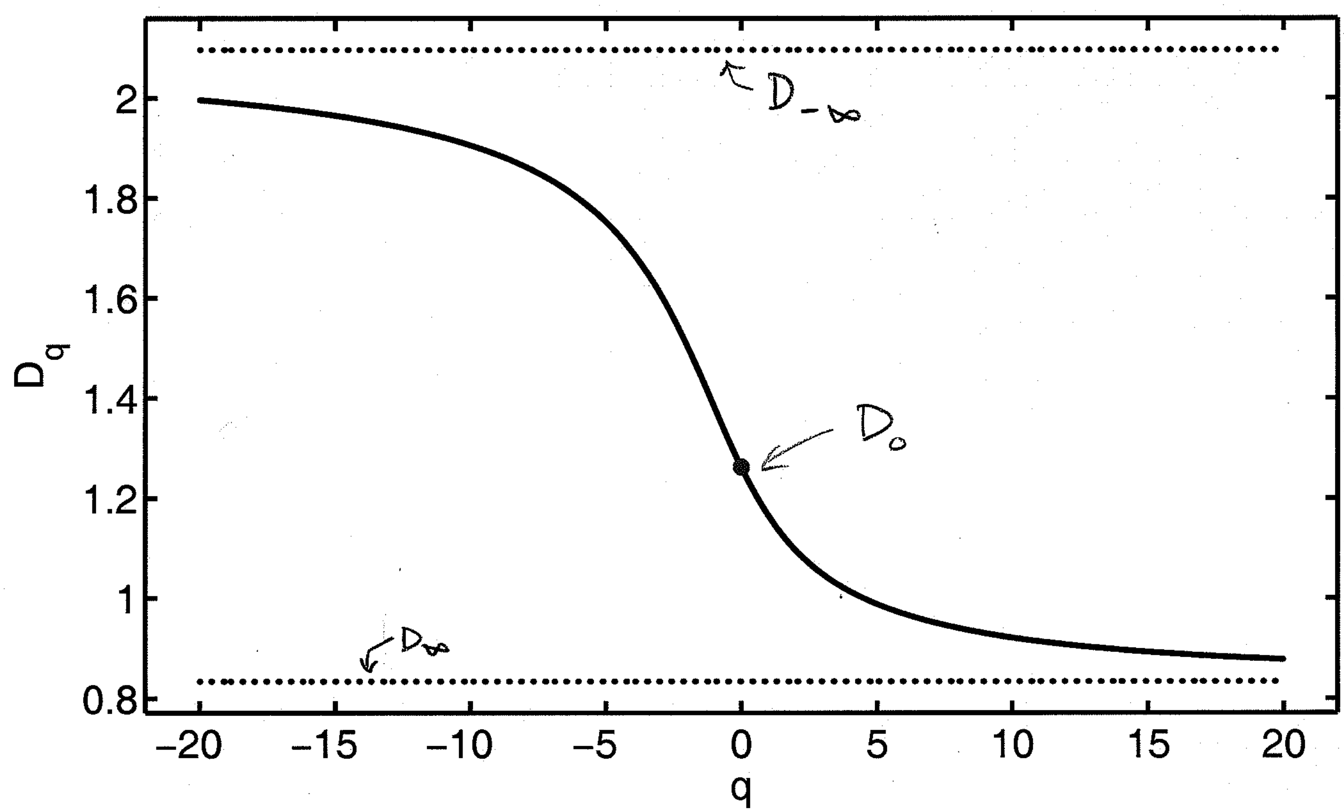
$$\boxed{\alpha(q) = -\frac{\alpha^q\log\alpha+\beta^q\log\beta+\gamma^q\log\gamma+\delta^q\log\delta}{(\alpha^q+\beta^q+\gamma^q+\delta^q)\log 3}} \quad (2)$$

Using (1) and (2), we can parametrically plot $(f(q),\alpha(q))$. See plot on next page.

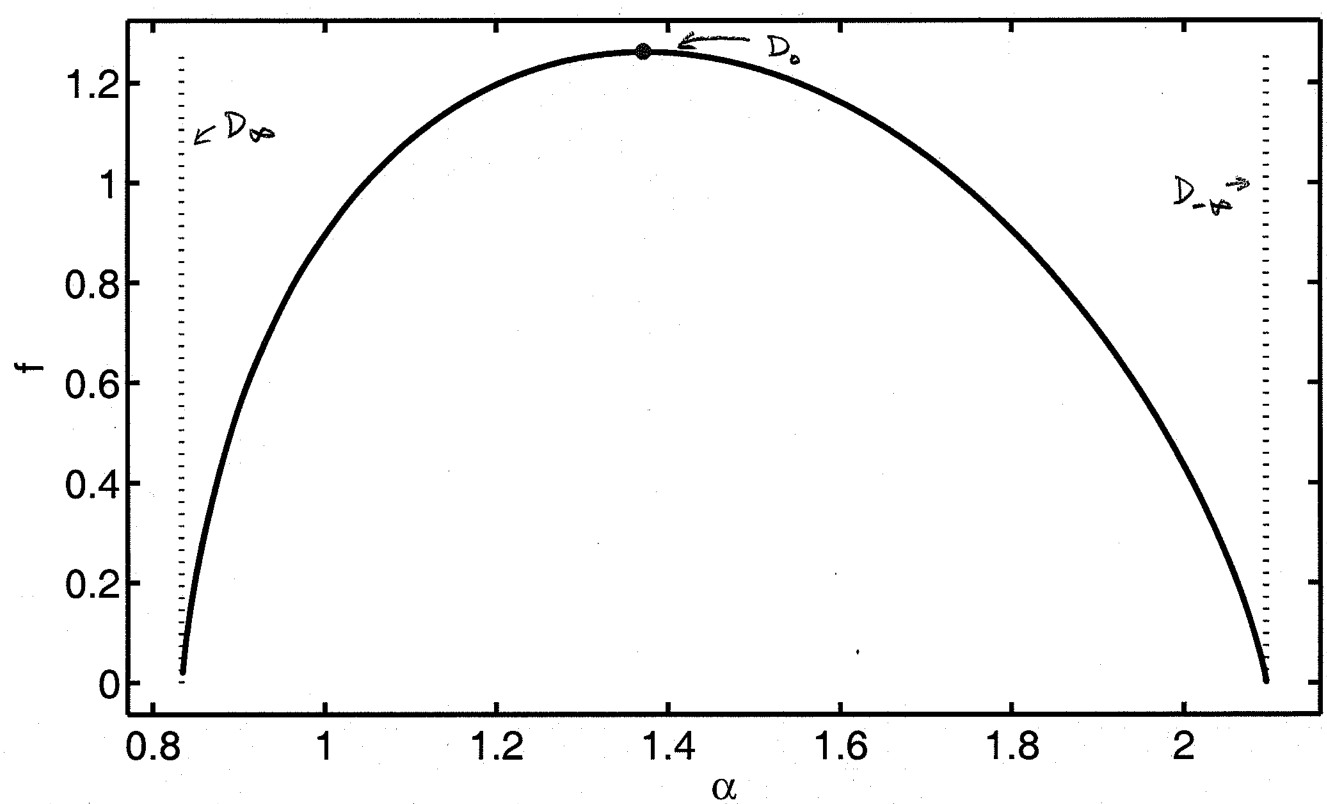
I chose $\alpha=0.4$, $\beta=0.3$, $\gamma=0.2$, $\delta=0.1$; the results change only quantitatively with ~~different~~ values (unless they're all equal).

① cont'd

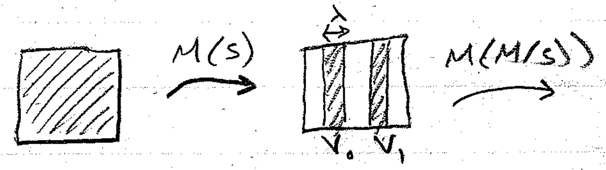
Dimension spectrum D_q

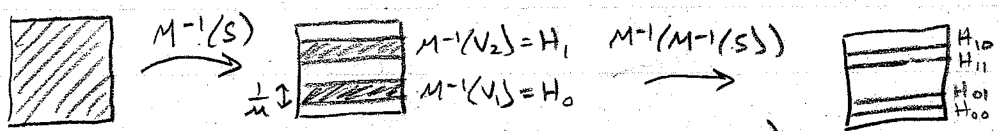


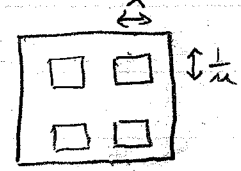
Multifractal spectrum $f(\alpha)$



② a)





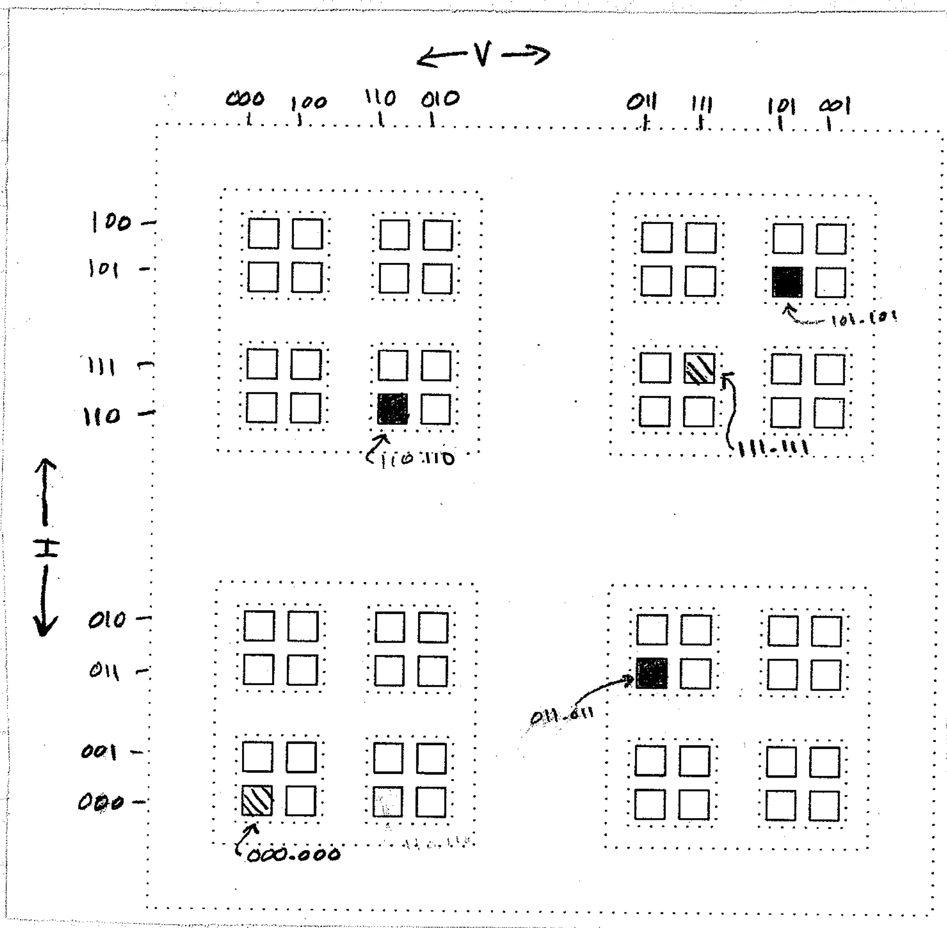


Invariant set is fractal with 4 boxes λ wide and $\frac{1}{\lambda}$ high, with 4 little boxes in each of them, etc. This forms generalized Cantor sets in $\hat{x}$ and $\hat{y}$. Each dimension can be calculated separately to get full dimension of invariant set (using similarity dimension):

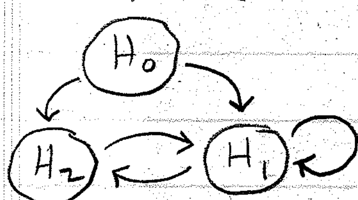
$$\boxed{d = d_x + d_y = \frac{\log 2}{\log(1/\lambda)} + \frac{\log 2}{\log(\mu)}}$$

b) On the next page, the fixed points (000.000, and 111.111) and the points in the period-3 orbit (011.011, 101.101, 110.110) are indicated. I located them to an accuracy of $\frac{1}{27}$ since this is the size of the cubes.

② b) cont'd



c) Based on the picture, $H_0 \rightarrow (H_1 \text{ or } H_2)$, $H_1 \rightarrow (H_1 \text{ or } H_2)$, and $H_2 \rightarrow H_1$.



period 1: $.\overline{1}$

period 2: $.\overline{12}$

period 3: $.\overline{112}$

period 4: $.\overline{1112}$