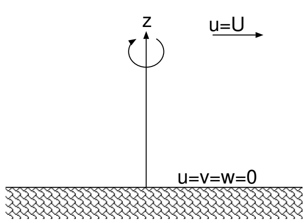
Homework #5
Introduction to physical oceanography

1. Material derivative for temperature: (i) Derive the Eulerian expression for the material derivative by considering the temperature rate of change following a particle that moves from (x,y,z) to $(x+\delta x, y+\delta y, z+\delta z)$ in a time δt. (ii) consider an Eulerian flow $u=(u,0,0)=(ax,0,0)$. Suppose that the temperature is given by $T=T(x)=cx$. Find the temperature as function of time for a particle that started at $x=x_0$ (this is the Lagrangian temperature following a fluid parcel). Find the time rate of change of this temperature, dT/dt in the Lagrangian picture. (iii) find the Eulerian material derivative of the temperature. are you getting the same result from the Eulerian and Lagrangian pictures? (iv) plot temperature versus x with the velocity field u(x) sketched as vectors right below the x-axis, and interpret physically what is happening (particularly, why the temperature is static in the eulerian frame but changing in the lagrangian frame?).

2. (i) Write the 3d momentum equations, including all terms derived in class in terms of the horizontal velocity vector $\vec{u}_H=(u,v)$ and the vertical velocity w, as well as in terms of $\nabla_H=(\frac{\partial}{\partial x},\frac{\partial}{\partial y})$ and $\frac{\partial}{\partial z}$. Do not use (u,v) at all in your answer, nor $\frac{\partial}{\partial x};\frac{\partial}{\partial y}$, nor the 3d ∇ or the 3d velocity vector $\mathbf{u}=(u,v,w)$. (ii) Do the same to the continuity equation (mass conservation) for incompressible fluid, and for the temperature equation. (iii) Explain what each term in each of these equations means physically and in which of the phenomena studied in class so far it plays a role. (iv) How many equations have you written? How many scalar equations are these equations actually equivalent to? How many scalar unknowns are there? What are these unknowns? Is the number of scalar/ vector equations equal to the number of scalar/ vector unknowns?

3. Write the velocity field for a solid body rotation $(u_r=0; u_\theta=\omega r)$ in Cartesian coordinates (x,y) instead of polar coordinates (u_r,u_θ). Use the expression for the curl in Cartesian coordinates to find the curl of these two vortices. Is the result the same as using the cylindrical coordinates used in class? Find the vorticity (curl) for an "irrotational" vortex $(u_r=0; u_q=\lambda/r)$ in both polar and Cartesian coordinates. what does the velocity field look like? why is it still called irrotational?

4. **Challenge problem:** Ekman Spiral: Show that the horizontal flow field in the Ekman layer forms a spiral: consider the case of a flow of a fluid bounded from below at $z=0$ by a rigid surface. At $z\to\infty$, the fluid velocity is uniform in the x direction: $\mathbf{u}(\mathbf{x},\mathbf{y},\mathbf{z}\to\infty)=(U,0,0)$. At the lower surface, $z=0$, the velocity must vanish: $(u,v,w)=(0,0,0)$.



Write the problem in terms of a deviation from the velocity at infinity, $\tilde{u}=u-U$; assume a balance between the Coriolis force and friction force due to vertical shear in the horizontal velocity: $-f\tilde{v}=A_v\partial^2\tilde{u}/\partial z^2$; $f\tilde{u}=A_v\partial^2\tilde{v}/\partial z^2$; assume a solution that is exponential in z and show that the solution for (u,v) is an exponentially decaying spiral (you can get some help from the section “The Ekman layer” in chapter 4 of Pedlosky’s book “Geophysical Fluid Dynamics”, see the course reading list).