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Introduction to Physical Oceanography
Homework 9 - Solutions

1. **Gravity waves in a finite depth water:**

   (a) Movie.

   (b) We have derived the equation of the motion and the boundary conditions in class. I will summarize the problem one more time. We assume that the motion is two dimensional in the $x-z$ plane where the waves are propagating in the x direction such that the equation of motion is given by the continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0 \qquad (1)$$

and assuming that the motion is irrotational (i.e. the vorticity is zero $\nabla\times\vec{u}=0$), the velocity can be written as the gradient of some potential ϕ such that

$$u=\frac{\partial\phi}{\partial x};\, w=\frac{\partial\phi}{\partial z}$$

the continuity equation is given by the Laplace equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}=0 \qquad (2)$$

We assume that the ocean has some depth equal to H. The boundary condition at the bottom $z=-H$ is simply no normal flow across the bottom of the ocean such that

$$w(z=-H)=\left.\frac{\partial\phi}{\partial z}\right|_{z=-H}=0 \qquad (3)$$

At the surface, we have a kinematic boundary condition which is the fluid particle cannot leave the surface

$$w=\frac{D\eta}{Dt}\approx\frac{\partial\eta}{\partial t}\Rightarrow\frac{\partial\phi}{\partial z}=\frac{\partial\eta}{\partial t} \qquad (4)$$

assuming that for small amplitude waves nonlinear terms are small and can be neglected. The second boundary condition at the surface is a dynamic boundary condition related to the fact the pressure at the surface is 0. This can be written using the Bernoulli's equation such that

$$\frac{\partial\phi}{\partial t}=-g\eta \qquad (5)$$

Assume that the solution to Laplace' equation is of the form

$$\phi=F(z)\cos(kx-\omega t) \qquad (6)$$

Plugging this solution into eq. 2, we get

$$\frac{D^2F(z)}{Dz^2} - k^2F(z) = 0 \Rightarrow F(z) = c_1e^{kz} + c_2e^{-kz} \tag{7}$$

Assume that $c_1 = 1$ and $c_2 = b$ (as given in this problem), the potential ϕ is then given by

$$\phi = \cos(kx - \omega t)(e^{kz} + be^{-kz}) \tag{8}$$

and the velocity components are

$$\begin{aligned} u &= \frac{\partial\phi}{\partial x} = -k\sin(kx-\omega t)(e^{kz}+be^{-kz}) \\ w &= \frac{\partial\phi}{\partial z} = k\cos(kx-\omega t)(e^{kz}-be^{-kz}) \end{aligned}$$

Using the boundary condition given by 3, we have

$$w(z=-H) = k\cos(kx-\omega t)(e^{-kH} - be^{kH}) = 0 \Rightarrow b = e^{-2kH} \tag{9}$$

Combining the boundary conditions 4 and 5 at the surface into a single equation leads to

$$\phi_{tt} = -g\phi_z \tag{10}$$

at the surface. By plugging 8 into the equation just derived, we get

$$-\omega^2\cos(kx-\omega t)(e^{kz}+be^{-kz}) = -gk\cos(kx-\omega t)(e^{kz}-be^{-kz}) \tag{11}$$

and using the value obtained for b

$$-\omega^2(1+b) = -g(1-b) \Rightarrow \omega^2\left(1+e^{-2kH}\right) = gk\left(1-e^{-2kH}\right) \tag{12}$$

This is the dispersion relation (frequency as function of the wavenumber) for gravity waves in an ocean of finite depth such that

$$\omega^2 = gk\frac{1-e^{-2kH}}{1+e^{-2kH}} \tag{13}$$

Using the following identities

$$sinh(z) = (e^z - e^{-z})/2\,;\, cosh(z) = (e^z + e^{-z})/2$$

the dispersion relation can be written as

$$\omega^2 = gk\tanh(kH) \Rightarrow \omega = \pm\sqrt{gk\tanh(kH)} \tag{14}$$

To summarize we have

$$\begin{aligned} \phi &= \cos(kx \pm gk\tanh(kH)t)(e^{kz} + e^{-2kH-kz}) \\ u &= -k\sin(kx \pm gk\tanh(kH)t)(e^{kz} + e^{-2kH-kz}) \\ w &= k\cos(kx \pm gk\tanh(kH)t)(e^{kz} - e^{-2kH-kz}) \end{aligned}$$

or

$$\begin{aligned} \phi &= 2e^{-kH}\cosh(k(H+z))\cos(kx \pm gk\tanh(kH)t) \\ u &= -2ke^{-kH}\cosh(k(H+z))\sin(kx \pm gk\tanh(kH)t) \\ w &= 2ke^{-kH}\sinh(k(H+z))\cos(kx \pm gk\tanh(kH)t) \end{aligned}$$

(c) Particle trajectories: to find the particle trajectories, we need to look at the Lagrangian coordinates (remember, the Lagrangian point of view is to follow a particle). Assume that a fluid particle is at rest at the point (x_0, z_0). Consider a small amplitude trajectory ξ, ζ around the point (x_0, z_0). This approximation of a small amplitude trajectory allow us to assume that the velocity field along this trajectory is nearly constant and equal to the velocity at the point (x_0, z_0). Therefore we can write

$$\begin{aligned} u &= \left.\frac{\partial \xi}{\partial t}\right|_{x_0, z_0} \\ w &= \left.\frac{\partial \zeta}{\partial t}\right|_{x_0, z_0} \end{aligned}$$

or

$$\begin{aligned} \left.\frac{\partial \xi}{\partial t}\right|_{x_0, z_0} &= -k\sin(kx_0 - \omega t)(e^{kz_0} + e^{-2kH - kz_0}) \\ \left.\frac{\partial \zeta}{\partial t}\right|_{x_0, z_0} &= k\cos(kx_0 - \omega t)(e^{kz_0} - e^{-2kH - kz_0}) \end{aligned}$$

leading to the trajectories

$$\begin{aligned} \xi &= -\frac{k}{\omega}\cos(kx_0 - \omega t)(e^{kz_0} + e^{-2kH - kz_0}) + \xi_0 \\ \zeta &= -\frac{k}{\omega}\sin(kx_0 - \omega t)(e^{kz_0} - e^{-2kH - kz_0}) + \zeta_0 \end{aligned}$$

We can write a single equation for the trajectory:

$$\begin{aligned} \left(\frac{\xi - \xi_0}{e^{kz_0} + e^{-2kH - kz_0}}\right)^2 &+ \left(\frac{\zeta - \zeta_0}{e^{kz_0} - e^{-2kH - kz_0}}\right)^2 \\ &= \frac{k^2}{\omega^2}\left(\cos^2(kx_0 - \omega t) + \sin^2(kx_0 - \omega t)\right) = \frac{k^2}{\omega^2} \end{aligned}$$

or

$$\left(\frac{\xi - \xi_0}{\frac{2ke^{-kH}}{\omega} cosh(kH + kz_0)}\right)^2 + \left(\frac{\zeta - \zeta_0}{\frac{2ke^{-kH}}{\omega} sinh(kH + kz_0)}\right)^2 = 1 \tag{15}$$

We obtained an equation for an ellipse where $\frac{2ke^{-kH}}{\omega} cosh(kH + kz_0)$ is the semi-major axis and $\frac{2ke^{-kH}}{\omega} sinh(kH + kz_0)$ the semi-minor axis. Figure 1 shows the trajectory of the particles at different depth. We can see that the axis of the ellipse get smaller as the depth increases, until the trajectory are just straight lines near the bottom.

2. The gradient of a scalar function $\phi(x, y, z)$ is given by

$$\begin{aligned} \nabla\phi &= \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k} \\ \nabla\phi &= (\phi_x, \phi_y, \phi_z) \end{aligned}$$

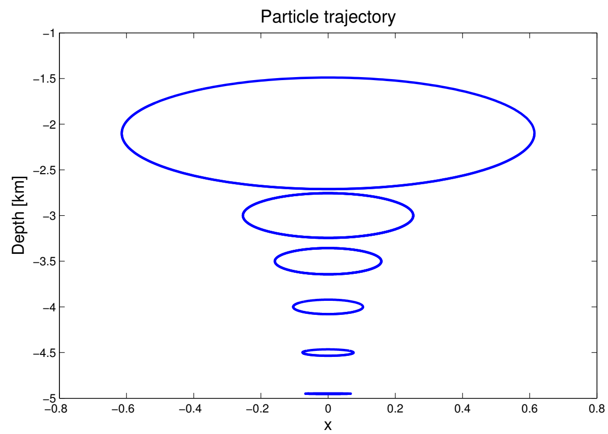

Figure 1: Trajectory of the particles in a finite depth ocean.

where $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ denote the unit vectors in the x, y, z directions respectively. The definition of the curl in 3D in Cartesian coordinates was given in the solutions to HW-07, so can just apply it:

$$\nabla \times \nabla \phi = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi_x & \phi_y & \phi_z \end{vmatrix}$$

$$\nabla \times \nabla \phi = ((\phi_z)_y - (\phi_y)_z)\hat{x} - ((\phi_z)_x - (\phi_x)_z)\hat{y} + ((\phi_y)_x - (\phi_x)_y\hat{z}) \tag{16}$$

Using the fact that $(\phi_z)_y = \phi_{zy} = \phi_{yz}$ and similarly for the others derivatives, we have

$$\begin{aligned} (\nabla \times \nabla \phi)_x &= \phi_{zy} - \phi_{yz} = \phi_{yz} - \phi_{yz} = 0 \\ (\nabla \times \nabla \phi)_y &= -\phi_{zx} + \phi_{xz} = -\phi_{xz} + \phi_{xz} = 0 \\ (\nabla \times \nabla \phi)_z &= \phi_{yx} - \phi_{xy} = \phi_{xy} - \phi_{xy} = 0 \end{aligned}$$

Each of the component of the vector $\nabla \times \nabla \phi$ is equal to 0, so that $\nabla \times \nabla \phi = 0$ for any scalar function ϕ.

3. **Vorticity in gravity waves:**

(a) In class, we found that for deep ocean gravity waves the velocity is given by

$$\begin{aligned} u &= -ak\sin(kx - \omega t)e^{kz} \\ w &= ak\cos(kx - \omega t)e^{kz} \end{aligned}$$

where $\omega = \pm\sqrt{gk}$.

The 3 components of the vorticity vector $\vec{\zeta} = \nabla \times \vec{u}$ are given by

$$\begin{aligned} \zeta_x &= w_y - v_z = 0 \\ \zeta_y &= u_z - w_x \\ \zeta_z &= v_x - u_y = 0 \end{aligned}$$

assuming that $v = 0$ and that there is no $y-$dependence (i.e. $\frac{\partial}{\partial y}$), the x and z component of the vorticity are zero and we are left with only the y component of the vorticity.

$$\zeta_y = u_z - w_x = -ak^2 \sin(kx - \omega t)e^{kz} + ak^2 \sin(kx - \omega t)e^{kz} = 0 \tag{17}$$

We've just shown that the three components of the vorticity vector are 0. Therefore our assumption of irrotational motion is justified.

(b) The left panel of figure 2 shows the velocity field at a given time using quiver.

(c) **challenge** The stream function ψ is given by

$$\begin{aligned} u &= \frac{\partial \psi}{\partial z} \\ w &= -\frac{\partial \psi}{\partial x} \end{aligned}$$

Using our expressions for u and w, we have (note: the stream function is evaluated at a given time, so that t is not a variable in the integration!)

$$\begin{aligned} \frac{\partial \psi}{\partial z} &= -ak \sin(kx - \omega t)e^{kz} \Rightarrow \psi(x,z) = -a \sin(kx - \omega t)e^{kz} + f(x,t) \\ \frac{\partial \psi}{\partial x} &= -ak \cos(kx - \omega t)e^{kz} \Rightarrow \psi(x,z) = -a \sin(kx - \omega t)e^{kz} + g(z,t) \end{aligned}$$

Obviously, $g(z,t) = f(x,t) = constant$ at a given time and this constant can be chosen arbitrarily to be zero. The stream function is then

$$\psi(x,z) = -a \sin(kx - \omega t)e^{kz} \tag{18}$$

The right panel of figure 2 shows the streamlines of the flow at a given time t_0.

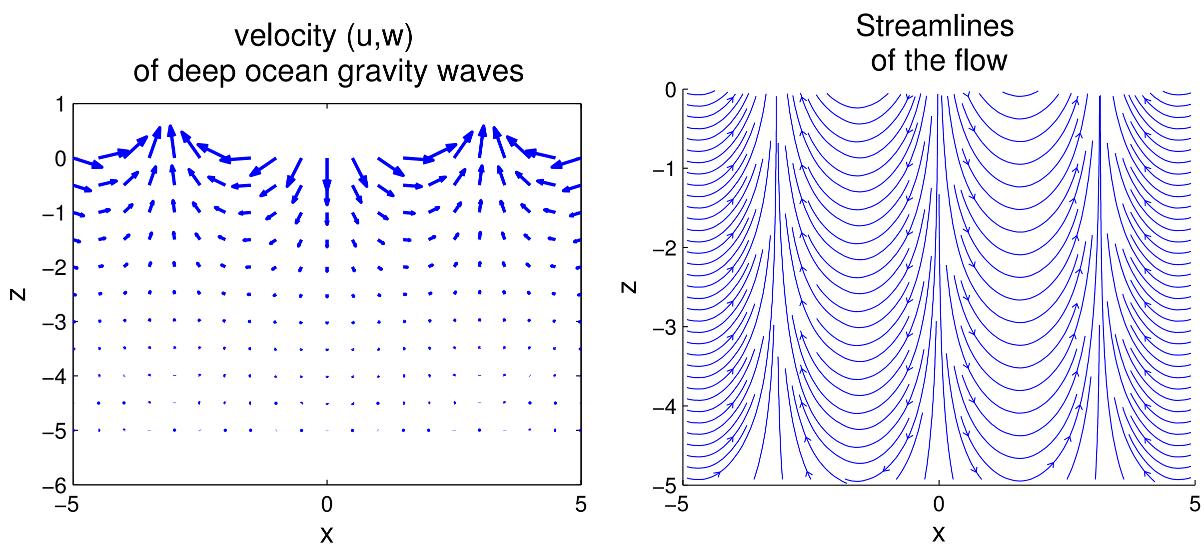

Figure 2: Left: Velocity field of deep ocean gravity waves; Right: Streamlines for deep ocean gravity waves.