Laure Zanna
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Introduction to Physical Oceanography
Homework 10 - Solutions

1. **Gravity waves in a finite depth water:**
Limit cases: the wavelength λ is equal to $\lambda = 2\pi/k$, where k is the wavenumber. The dispersion relation is given by

$$\omega = \pm\sqrt{gk \tanh(kH)} \quad (1)$$

1. Shallow water: In this limit, the depth of the ocean is small compared to the wavelength $H << \lambda$ or $H << k^{-1}$. For this limit we have

$$\tanh(kH) \approx kH \quad (2)$$

such that

$$\omega \approx k\sqrt{gH} \quad (3)$$

The gravity waves in shallow water are non-dispersive waves so that the phase and group velocity are equal and given by

$$c = \sqrt{gH} \quad (4)$$

2. Deep water: the depth of the ocean is large compared to the wavelength $H >> \lambda$ or $H >> k^{-1}$. Using this limit, we obtain

$$\tanh(kH) \approx 1 \quad (5)$$

such that

$$\omega \approx \sqrt{gk} \quad (6)$$

For deep gravity waves, the phase velocity is given by

$$c_{phase} = \frac{\omega}{k} = \sqrt{g/k} \quad (7)$$

and the group velocity by

$$c_{group} = \frac{\partial\omega}{\partial k} = \frac{1}{2}\sqrt{g/k} \quad (8)$$

Both limits are similar to the ones obtained in class.

2. **Group and phase velocities:**

   (a) Using the dispersion relation for deep gravity waves $\omega = \sqrt{gk}$ for 2 different wave numbers $k = 0.9$ and $k' = 1.1$, the frequencies obtained are $\omega = 2.9714\,sec^{-1}$ and $\omega = 3.2850\,sec^{-1}$ respectively.

   (b) See Figure 1

   (c) See Figure 1

(d) From figure 1, the phase velocity is

$$c_{ph} = \frac{\Delta x_{black}}{\Delta t} = \frac{21.25}{2.0086} \approx 10.57\, m/s$$

The group velocity is

$$c_g = \frac{\Delta x_{red}}{\Delta t} = \frac{9.9}{2.0086} \approx 4.9288\, m/s$$

As expected we've found that the group velocity is roughly half the phase velocity such that

$$c_g = \frac{1}{2} c_{ph} \tag{9}$$

3. **Phase velocities in 2d:**

(a) See figure 2

(b) See figure 2

(c) The phase velocity in the x-direction is given by

$$c_x = \frac{\omega}{k}$$

such that $\omega = 2\pi/T$ and T is the period. The phase velocity in the y-direction is given by

$$c_y = \frac{\omega}{l}$$

The ratio between the phase velocity in the $x-$direction and the $y-$direction is

$$\frac{c_x}{c_y} = \frac{l}{k} \tag{10}$$

Using $k = 5l$, the ratio is

$$\frac{c_x}{c_y} = \frac{1}{5}$$

such that the phase velocity in the $x-$ direction should be 5 times smaller than the phase velocity in the $y-$ direction. Let's find these values from the plot. We need to measure the distance between the 2 contours plotted at 2 different times on the x and y axis. I found $\Delta x = 0.2514$ and $\Delta y = 1.2567$ such that

$$c_x = \frac{\Delta x}{\Delta t} = \frac{0.2514}{0.0562} \approx 4.4744\, m/s \tag{11}$$

and

$$c_y = \frac{\Delta y}{\Delta t} = \frac{1.2567}{0.0562} \approx 22.3666\, m/s \tag{12}$$

We have found that c_y is 5 times bigger than c_x as expected.

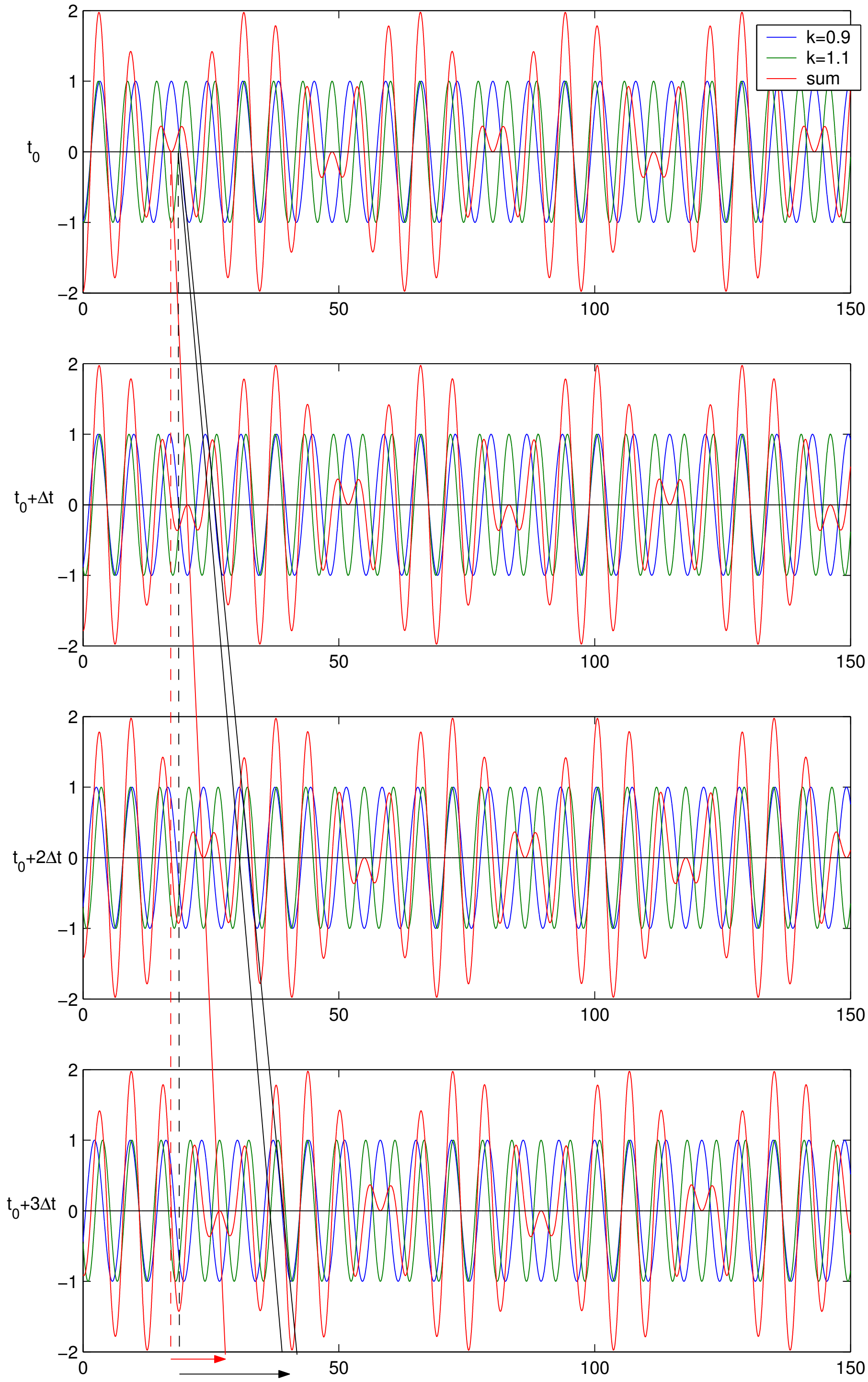

Figure 1: Two deep gravity waves (with slightly different wave numbers) and their sum.

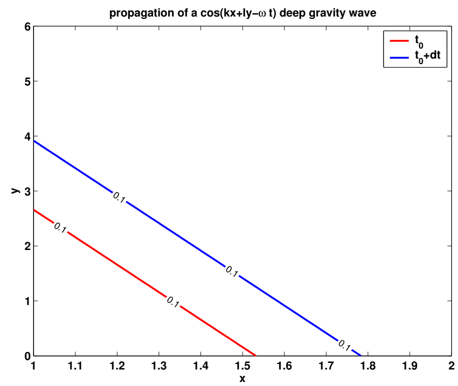

Figure 2: Contour of sea surface elevation for deep gravity waves at 2 consecutive times.

4. **Internal Kelvin and Rossby waves**

The dispersion relation for internal Kelvin wave is given by

$$\omega = K\sqrt{g'H}$$

and the dispersion relation for internal Rossby waves by

$$\omega = -\frac{\beta k}{K^2 + L_R^{-2}}$$

where $L_R = \sqrt{g'H}/f$ is the internal Rossby radius of deformation and $K^2 = k^2 + l^2$.

(a) Figure 4 shows the frequency as function of the zonal wave number k for $l = 100\,km$.

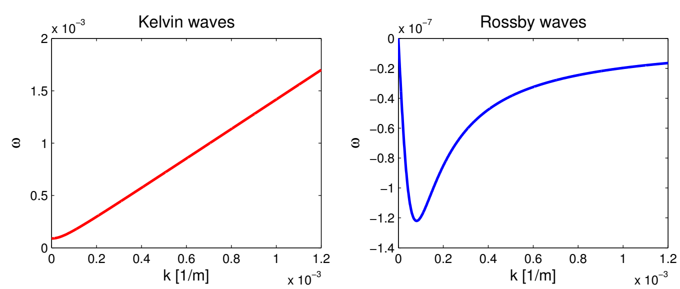

Figure 3: Frequency as function of k for internal Kelvin Waves (left) and internal Rossby waves (right).

(b) To make my like easier, I will take $l = 0$. The phase velocity c_{ph} is the velocity at which a single wave is traveling and is given by

$$c_{ph} = \frac{\omega}{k} \tag{13}$$

The group velocity c_g is the velocity at which the envelope is propagating and is given by

$$c_g = \frac{\partial \omega}{\partial k} \tag{14}$$

For Kelvin waves, we have $\omega = k\sqrt{g'H}$, such that

$$c_{ph}^{Kelvin} = c_g^{Rossby} = \sqrt{g'H} \tag{15}$$

For Rossby waves (which are dispersive waves), we have $\omega = -\frac{\beta k}{k^2+L_R^{-2}}$, such that

$$c_{ph}^{Rossby} = -\frac{\beta}{k^2+L_R^{-2}} \tag{16}$$

and

$$c_g^{Rossby} = \frac{\beta\left(k^2-L_R^{-2}\right)}{\left(k^2+L_R^{-2}\right)^2} \tag{17}$$

For wavelength taken between 10 to 10000km, we have a range of wavenumber between $10^{-3}\,m^{-1}$ and $10^{-8}\,m^{-1}$.

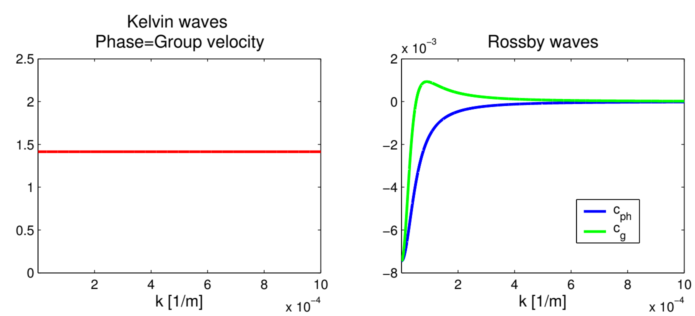

Figure 4: Phase and Group velocity as function of k for internal Kelvin waves (left) and internal Rossby waves (right).

(c) A storm passing over the Pacific ocean excites internal waves with a wavelength of $\lambda = 2000\,km$. Let consider the frequency, phase and group velocity for the different types of waves excited by this storm. I will take $l = 0$, $\theta_0 = 30°N$, $g' = 0.02\,m/s^2$ and $H = 100\,m$.

(i) Poincare waves: The frequency is $\omega = \sqrt{f^2+g'Hk^2} \approx 7.3135 \cdot 10^{-5}\,s^{-1}$. For long wave (or small wave number), the frequency of Poincare waves is dominated by the Coriolis frequency. The phase velocity is

$$c_{ph} = \sqrt{\frac{f^2}{k^2}+g'H} \approx 23.2796\,m/s$$

and the group velocity

$$c_g = \frac{gHk}{\sqrt{\frac{f^2}{k^2}+g'H}} \approx 0.0859\,m/s$$

The group velocity for long Poincare waves is much smaller than the phase velocity.
(ii) Kelvin waves: The frequency is $\omega = k\sqrt{g'H} \approx 4.4429 \cdot 10^{-6}\, s^{-1}$, and the phase and group velocity are $c \approx 1.4142\, m/s$.
(iii) Rossby waves: The frequency $\omega = -\frac{\beta k}{k^2 + L_R^{-2}} \approx 2.3313 \cdot 10^{-8}\, s^{-1}$ (with $\beta \approx 2 \cdot 10^{-11}\, m^{-1}$, $L_R \approx 20\, km$) . The phase velocity is westward (always!) and equal to $c_{ph} \approx -0.0074\, m/s$. The group velocity is $c_g = -0.0074\, m/s$. For such long waves, we obtained that the group and phase velocity are equal and both westward.
The Pacific ocean is roughly $L = 15000\, km$ wide, such that the time for Kelvin and Rossby wave to cross the Pacific is equal to

$$\Delta t = \frac{L}{c} \tag{18}$$

where c is the velocity of Kelvin or Rossby waves. For the Kelvin wave, I've found $\Delta t \approx 4$ months while for the Rossby waves, I found $\Delta t \approx 778$ months.
Note: the Rossby waves are really slow... if we were considering a storm exciting these waves at $10°N$ instead of $30°N$, they will be 10 times faster. In addition, if the depth of the upper was layer was $1000\, m$ instead of $100\, m$, the speed will be again 10 times faster. So, if we take $H = 1000\, m$ and $\theta_0 = 10°N$, we will get a speed of $0.6 m/s$ and the time for the Rossby wave to cross the Pacific will be around 8 months.

5. **Group velocities in 2d:**

(a) The contour plots of sea surface elevation at time t_0 and $t_0 + dt$ for the train of deep gravity waves given by

$$\eta(x,y,t) = \cos(kx + ly - \omega t) + \cos((k+\delta)x + (l+\delta)y - (\omega + \delta\omega)t) \tag{19}$$

are shown in figure 5.

(b) The group velocity in the direction of propagation is found by measuring the distance traveled by the envelope in the amount of time Δt.

(c) The upper panel of figure 6 shows the amplitude at $y = 0$ as function of x while the lower panel of this figure shows the amplitude at $x = 0$ as function of y both at $t = t_0$. Super-imposed on figure 6, we have the amplitude at $t = t_0 + \Delta t$.

(d) The group velocity from the contour plot in the direction of propagation is

$$c_g = \frac{10.5}{0.1 \cdot 2\pi/\omega} = 45.7755\, m/s$$

(e) The group velocity in the x-direction is

$$c_g^x = \frac{\Delta x}{\Delta t} = \frac{8.95}{0.1 \cdot 2\pi/\omega} \approx 39.0181\, m/s$$

The group velocity in the y-direction is

$$c_g^y = \frac{\Delta y}{\Delta t} = \frac{5.05}{0.1 \cdot 2\pi/\omega} \approx 22.0158\, m/s$$

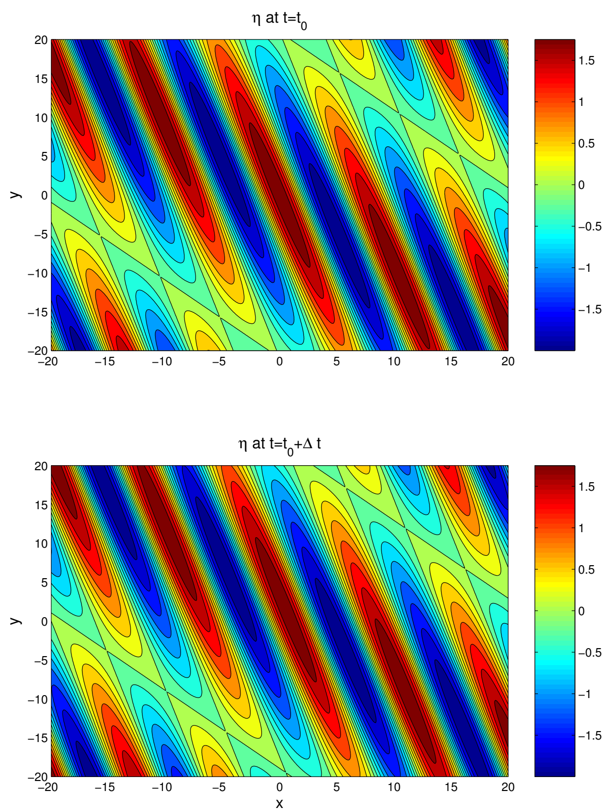

Figure 5: Contours of sea surface elevation η at 2 different times.

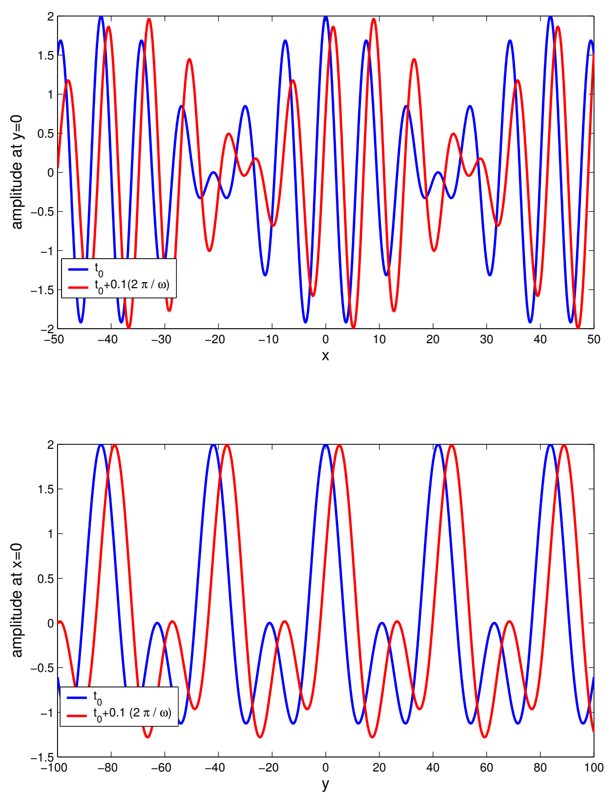

Figure 6: Amplitude at $y = 0$ as function of x (top panel); amplitude at $x = 0$ as function of y (bottom panel).

(f) The group velocity in the direction of propagation is therefore given by

$$c_g = \sqrt{(c_g^x)^2 + (c_g^y)^2} = 44.8008m/s \tag{20}$$

We can see that the x and the y component of the group velocity allow us to obtain the group velocity is the direction of propagation, as we expect it from a vector such that

$$\vec{c}_g = \vec{c}_g^x \hat{x} + \vec{c}_g^y \hat{y}$$

It is not the case from the phase velocity where

$$\vec{c}_{ph} \neq \vec{c}_{ph}^x \hat{x} + \vec{c}_{ph}^y \hat{y}$$

as seen in question 3. As explained in class, the energy is carried by the wave packet (by the envelope formed by the combination of each of the waves) and moves with the group velocity. It makes sense physically that the group velocity, at which a physical quantity like energy is propagating, is a vector.