Equatorial Kelvin waves

The derivation here follows Gill (1982). Consider first the case of an equatorial Kelvin wave, which is a special solution for the case of zero meridional velocity ($v = 0$), no forcing and no dissipation. The ocean is composed of an upper moving layer with a thickness $h(x, y, t)$ and a mean thickness H and a lower, resting layer. The thickness of the upper layer varies due to variations in the interface between the two layers. The two layer densities are ρ_1 and ρ_2, and the reduced gravity which appears in the equations is $g' = g(\rho_2 - \rho_1)/\rho_0$. The momentum and linearized mass conservation equations are then,

$$\begin{aligned} \frac{\partial u}{\partial t} - fv &= -g'\frac{\partial h}{\partial x} \\ \frac{\partial v}{\partial t} + fu &= -g'\frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0 \end{aligned}$$

with $f = f_0 + \beta y$ and $f_0 = 0$, reduce to,

$$\begin{aligned} \frac{\partial u}{\partial t} &= -g'\frac{\partial h}{\partial x} \\ \beta y u &= -g'\frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial t} + H\frac{\partial u}{\partial x} &= 0. \end{aligned}$$

Note the geostrophic balance in the y-momentum equation. Substituting $e^{i(kx-\omega t)}$ dependence for all three variables, we get from the first that $u = (kg'/\omega)h$, so that the third one gives the dispersion relation

$$\omega^2 = (g'H)k^2$$

which is the dispersion relation of a simple shallow water gravity wave. The second equation then gives $\beta y \frac{kg'}{\omega} h = -g' \frac{\partial h}{\partial y}$, or

$$\frac{\partial h}{\partial y} = -\frac{\beta k}{\omega} y h.$$

We are searching for equatorial-trapped solutions, and we note that the solution for the y-structure decays away from the equator only when $k > 0$. This implies that the wave solution we have found must be eastward propagating! Using the dispersion relation, with

$$\mathrm{c} \equiv \sqrt{\mathrm{g'H}} \approx (9.8 \times 10^2 \ \mathrm{cm\,sec^{-2}} \times 5 \times 10^{-3} \times 100 \times 10^2 \ \mathrm{cm})^{1/2} \approx 2.2 \ \mathrm{m\,sec^{-1}}$$

we finally have

$$h_{Kelvin}(x, y, t) \propto e^{-\frac{1}{2}(\beta/c)y^2} e^{i(kx-\omega t)}.$$

Note that the decay scale away from the equator is the equatorial Rossby radius of deformation defined as

$$\mathrm{L_{eq}^R} \equiv \sqrt{\mathrm{c}/2\beta} \approx (\mathrm{c}/(2 \times 2.3 \times 10^{-11} \ \mathrm{m^{-1}sec^{-1}}))^{1/2} \approx 220 \ \mathrm{km}$$

References

Gill, Adrian E. (1982). *Atmosphere-Ocean Dynamics*. Academic Press, Inc, San Diego, CA, 662pp, p. 662.