External (Equilibrium) Tide

ChatGPT

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1 Deriving the degree-2 tide-generating potential at the ocean surface

Setup. Let O be Earth's center, $\mathbf{r}$ the position of a surface point ($|\mathbf{r}| = R$), and $\mathbf{D}$ the vector from O to the perturbing body (Moon/Sun), with $|\mathbf{D}| = d$. Let ψ be the angle between $\mathbf{r}$ and $\mathbf{D}$, so $\cos\psi = \hat{\mathbf{r}}\cdot\hat{\mathbf{D}}$.

Gravitational potential and expansion. The body's potential at $\mathbf{r}$ is

$$\Phi(\mathbf{r}) = -\frac{GM'}{|\mathbf{D}-\mathbf{r}|}.$$

For $r \ll d$ (true for Moon/Sun), expand with Legendre polynomials:

$$\frac{1}{|\mathbf{D}-\mathbf{r}|} = \frac{1}{d}\sum_{n=0}^{\infty}\left(\frac{r}{d}\right)^n P_n(\cos\psi).$$

Why Legendre polynomials? Choose coordinates with the z-axis along $\mathbf{D}$. Then

$$|\mathbf{D}-\mathbf{r}| = \sqrt{d^2 + r^2 - 2dr\cos\psi} \quad \Rightarrow \quad \frac{1}{|\mathbf{D}-\mathbf{r}|} = \frac{1}{d}\left(1 - 2\frac{r}{d}\cos\psi + \frac{r^2}{d^2}\right)^{-1/2}.$$

For $|t| < 1$, the generating function for Legendre polynomials is

$$(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n.$$

Set $x = \cos\psi$ and $t = r/d$ (note $r/d < 1$ for the ocean point and lunar/solar distances). This yields the Laplace (Legendre) expansion

$$\boxed{\frac{1}{|\mathbf{D}-\mathbf{r}|} = \frac{1}{d}\sum_{n=0}^{\infty}\left(\frac{r}{d}\right)^n P_n(\cos\psi)}.$$

Convergence. Because $|P_n(x)| \le 1$ for $|x| \le 1$, the remainder after N terms obeys

$$\left|\frac{1}{|\mathbf{D}-\mathbf{r}|} - \frac{1}{d}\sum_{n=0}^{N}\left(\frac{r}{d}\right)^n P_n(\cos\psi)\right| \le \frac{1}{d}\sum_{n=N+1}^{\infty}\left(\frac{r}{d}\right)^n = \frac{1}{d}\,\frac{(r/d)^{N+1}}{1 - r/d},$$

so the series converges absolutely and uniformly on any set with $r \le R < d$.

Remove irrelevant terms. Tides depend on *relative* forces inside the Earth-ocean system. Subtract:

- the constant term (no forces): $-GM'/d$,
- the uniform-acceleration (dipole) term, which is canceled by the Earth's free-fall toward the body:

$$\nabla\Phi(0)\cdot\mathbf{r} = +\frac{GM'}{d^3}(\mathbf{D}\cdot\mathbf{r}) = +\frac{GM'}{d}\left(\frac{r}{d}\right)P_1(\cos\psi).$$

The remainder is the **tide-generating potential** (TGP):

$$U(\mathbf{r}) \;\equiv\; -GM'\left[\frac{1}{|\mathbf{D}-\mathbf{r}|} - \frac{1}{d} - \frac{\mathbf{D}\cdot\mathbf{r}}{d^3}\right] = -\frac{GM'}{d}\sum_{n=2}^{\infty}\left(\frac{r}{d}\right)^n P_n(\cos\psi).$$

Keep the leading term (quadrupole). The dominant piece is $n = 2$:

$$\boxed{U(\mathbf{r}) \;\approx\; -\frac{GM'r^2}{d^3}\,P_2(\cos\psi)}, \qquad P_2(x) = \tfrac{1}{2}(3x^2-1).$$

By the common oceanographic sign convention (positive "upward" height when we later divide by g), we take

$$\boxed{U(\phi,\lambda,t) = \frac{GM'R^2}{d(t)^3}\,P_2(\cos\psi)}, \quad \cos\psi = \sin\phi\,\sin\delta(t) + \cos\phi\,\cos\delta(t)\cos H(t),$$

evaluated at the **ocean surface** $r = R$. Here $\delta(t)$ is the body's declination and $H(t)$ its local hour angle.

(Numbers: for the Moon, $GM'R^2/(gd^3) \approx 0.357$ m; for the Sun ≈ 0.164 m. These set the equilibrium height scale before Love-number and geometric factors.)

2 Start from the tidal potential

For a perturber of mass M' (Moon or Sun) at distance $d(t)$, the degree-2 tidal potential at the ocean surface (Earth radius R) is

$$U(\phi,\lambda,t) = \frac{GM'R^2}{d(t)^3}\,P_2(\cos\psi),$$

with ϕ latitude, λ longitude, and

$$\cos\psi = \sin\phi\,\sin\delta(t) + \cos\phi\,\cos\delta(t)\cos H(t).$$

Here $\delta(t)$ is the body's declination and $H(t)$ its local hour angle. Using $P_2(x) = \frac{1}{2}(3x^2-1)$ and expanding, one gets the standard harmonic split:

$$\boxed{\begin{aligned} P_2(\cos\psi) = {} & \underbrace{\tfrac{1}{4}\left(3\sin^2\phi-1\right)\left(3\sin^2\delta-1\right)}_{\text{zonal (m=0)}} \\ & + \underbrace{\tfrac{3}{4}\,\sin 2\phi\,\sin 2\delta\;\cos H}_{\text{diurnal (m=1)}} \\ & + \underbrace{\tfrac{3}{4}\,\cos^2\phi\,\cos^2\delta\;\cos 2H}_{\text{semidiurnal (m=2)}} \end{aligned}}$$

3 Equilibrium sea level (external tide)

The sea surface is (to first order) an equipotential of **gravity + tidal potential + Earth deformation**. With Love numbers k_2 (potential) and h_2 (vertical) the equilibrium surface height relative to the solid Earth is

$$\eta_{\rm eq} = \kappa\,\frac{U}{g}, \qquad \kappa \equiv 1 + k_2 - h_2 \approx 0.69,$$

so the **amplitude** of any harmonic component is $\kappa\,U/g$ times its geometric coefficient.

Plugging the expansion above into $\eta_{\rm eq}$ and keeping only the time-varying parts (the zonal $m = 0$ piece is a static offset), we get the diurnal and semidiurnal harmonic envelopes:

- **Diurnal (m=1):** coefficient $\frac{3}{4}\sin 2\phi\,\sin 2\delta$ multiplying $\cos H$.
- **Semidiurnal (m=2):** coefficient $\frac{3}{4}\cos^2\phi\,\cos^2\delta$ multiplying $\cos 2H$.

4 Read off the four A_i

Let

$$\mathcal{C}_L(t) \equiv \kappa\,\frac{GM_{\sf Moon}R^2}{g\,d_{\sf Moon}(t)^3}, \qquad \mathcal{C}_S(t) \equiv \kappa\,\frac{GM_{\sf Sun}R^2}{g\,d_{\sf Sun}(t)^3}.$$

(Numerically, at mean distances: $\mathcal{C}_L \approx 0.357 \times 0.69 \approx 0.246$ m and $\mathcal{C}_S \approx 0.246 \times 0.459 \approx 0.113$ m; the extra $\frac{3}{4}$ below gives the harmonic's height scale.)

Semidiurnals

$$\boxed{\begin{aligned} A_{\rm M2}(\phi,t) &= \tfrac{3}{4}\,\mathcal{C}_L(t)\ \cos^2\!\phi\ \cos^2\!\delta_m(t), \\ A_{\rm S2}(\phi,t) &= \tfrac{3}{4}\,\mathcal{C}_S(t)\ \cos^2\!\phi\ \cos^2\!\delta_s(t). \end{aligned}}$$

These are the instantaneous **equilibrium** amplitudes of the lunar and solar semidiurnal parts. At the equator with $\delta \approx 0°$: $A_{\rm M2} \approx 0.185$ m, $A_{\rm S2} \approx 0.085$ m; their ratio is ≈ 0.46 from $(M_S/d_S^3)/(M_L/d_L^3)$.

Diurnals

The diurnal equilibrium tide contains separate lunar and solar pieces that share the same latitude factor $\sin 2\phi$ but different declinations. Grouping them into the classical constituents:

- **Lunar diurnal (O1):**

$$\boxed{A_{\rm O1}(\phi,t) = \tfrac{3}{4}\,\mathcal{C}_L(t)\ |\sin 2\phi|\ |\sin 2\delta_m(t)|.}$$

- **Lunisolar diurnal (K1):** is the **sum** of lunar and solar diurnal forcings at (nearly) the same frequency,

$$\boxed{A_{\rm K1}(\phi,t) = \tfrac{3}{4}\,|\sin 2\phi|\ \Big|\ \mathcal{C}_L(t)\ \sin 2\delta_m(t)\ +\ \mathcal{C}_S(t)\ \sin 2\delta_s(t)\ \Big|.}$$

Notes

- $\delta_m(t)$ and $\delta_s(t)$ are the Moon/Sun declinations; $d_{\sf Moon}, d_{\sf Sun}$ their distances.

- The absolute-value bars simply indicate that “amplitude” is nonnegative; the signs are carried by the phases.
- In full astronomical practice, each A_i is multiplied by a slowly varying **nodal factor** $f_i(t)$ and phase correction $u_i(t)$ (18.6-yr modulation for lunar terms), and split into nearby lines (e.g., $M_2, N_2, \dots; K_1, P_1, O_1, Q_1, \dots$).

5 From equilibrium to real ocean amplitudes

The **observed external tide** at a location (ϕ, λ) is the equilibrium forcing above passed through the ocean’s **dynamical admittance** $\mathcal{G}_i(\phi, \lambda)$ (solution of the Laplace tidal equations with continents, depth, friction, SAL, etc.):

$$\eta_i(\phi, \lambda, t) = \Re\Big\{\mathcal{G}_i(\phi, \lambda)\, A_i(\phi, t)\, e^{-i\omega_i t + i\varphi_i}\Big\}, \qquad \boxed{|\,\eta_i| = |\mathcal{G}_i|\, A_i.}$$

This is why actual A_i maps show strong regional structure and shelf/basin resonances.