ENSO teleconnections

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Motivation. Our motivation is the ENSO teleconnection which lead to worldwide weather effects. These are propagated from the equatorial pacific via equivalent-barotropic atmospheric Rossby waves. The waves lead to sequences of high and low pressure centers that affect remote weather. This is demonstrated in results of barotropic model runs, say Fig. 3 from Hoskins and Karoly (1981), showing global propagation of waves due to tropical disturbances.

Barotropic Rossby waves in a zonally-symmetric mean zonal flow. Start with a derivation of PV conservation as an equation for a Rossby wave in presence of a mean zonal flow. In Cartesian coordinates, the barotropic QG potential vorticity conservation is $(\partial_t + u\partial_x + v\partial_y)(v_x - u_y + \beta y) = 0$; let $u = U(y) + u'$, $v = v'$, and linearize, to find the linearized quasi-geostrophic vorticity equation (QG PV), $(\partial_t + U\partial_x)(v'_x - u'_y) + (\beta - U_{yy})v' = 0$. Introducing a stream function $v' = \psi_x$, $u' = -\psi_y$ and effective beta $\beta_{\text{eff}} = \beta - U_{yy}$ to find the final wave equation,

$$\mathcal{L}\psi = (\partial_t + U\partial_x)\nabla^2\psi + \beta_{\text{eff}}\psi_x = 0.$$

These waves travel large distances so it is important to keep track of spherical coordinates. Hoskins and Karoly (1981) therefore use the same equation, but in spherical coordinates using the Mercator projection (their equations 5.1–5.16). This leads to the same QG PV equation, except replacing β_{eff} with their β_M and of the mean zonal wind U with their u_M, both defined as,

$$\begin{aligned}\beta_M &= \frac{2\Omega}{a} - \frac{d}{dy}\frac{1}{\cos^2\phi}\frac{d}{dy}(\cos^2\phi\, u_M)\\ u_M &= U/\cos\phi.\end{aligned}$$

Here Ω is the Earth rotation rate and ϕ is latitude, corresponding to the y coordinate. We now substitute a wave solution for the stream function, $\psi = e^{i(kx+ly-\omega t)}$ into the above linearized potential vorticity equation. The resulting Rossby wave dispersion relation is,

$$\omega = u_M(y)k - \frac{\beta_M(y)k}{k^2 + l^2} = \bar{\Omega}(k, l, y).$$

The final functional form on the RHS emphasizes that the frequency is a function of the wavenumbers and of latitude. The consequences of this will become clear shortly.

Ray tracing reminder. Consider a wave function $\psi(x,y,t) = e^{i\phi(x,y,t)}$ satisfying the wave equation $\mathcal{L}\psi = 0$. E.g., the above QG PV equation, or $\mathcal{L}\psi = \psi_{tt} - c^2(x,t)\psi_{xx} = 0$ (note that the wave velocity $c(x,t)$ is assumed in this example to be a function of space and time, and therefore so would be the dispersion relation below. Taylor-expand the phase $\phi \approx (\nabla\phi)\mathbf{x} + (\phi_t)t$, and define $\mathbf{k} \equiv \nabla\phi$, $\omega \equiv -\phi_t$. Because the order of spatial and temporal derivatives may be exchanged, we immediately have $\partial_t \nabla\phi = \nabla\partial_t\phi$, which implies,

$$\frac{\partial \mathbf{k}}{\partial t} + \nabla\omega = 0. \tag{1}$$

The wave equation gives the dispersion relation $\mathcal{L}(\omega, \mathbf{k}, \mathbf{x}, t) = 0$. For the above simple wave equation this means, for example $-\omega^2 + c^2(x,t)k^2 = 0$. Generally, we can write this as,

$$\omega = \bar{\Omega}(k, l, x, y, t). \tag{2}$$

For the above example, this takes the form $\omega^2 = \bar{\Omega}^2(k, x, t) = c^2(x,t)k^2$. Now, taking the spatial gradient of the frequency, we have

$$\nabla\omega = \frac{\partial\bar{\Omega}}{\partial\mathbf{k}} \cdot \nabla\mathbf{k} + \nabla\bar{\Omega},$$

substituting into (1),

$$\frac{\partial\mathbf{k}}{\partial t} + \mathbf{c_g} \cdot \nabla\mathbf{k} = -\nabla\bar{\Omega}. \tag{3}$$

Differentiating (2) wrt t, and using (1),

$$\frac{\partial\omega}{\partial t} + \mathbf{c_g} \cdot \nabla\omega = \bar{\Omega}_t. \tag{4}$$

The above two equations (3, 4) provide the rate of change of the wavenumber and frequency following a path set by the group velocity. The ray path follows the group velocity, and is therefore given by,

$$\frac{d\mathbf{x}}{dt} = \mathbf{c_g}. \tag{5}$$

These final three equations are the wave tracing equations, allowing us to track the location of a wave package traveling at the wave group velocity, as well as its wavenumber and frequency, which change depending on changes to the medium in which the wave propagates.

Consequences for zonally-symmetric, steady, mean zonal flow. Because the mean flow u_M and therefore the dispersion relation are x and t-independent, we have k =constant and ω =constant along a ray, while $d_g l/dt \equiv (\partial_t + \mathbf{c}_g \cdot \nabla)l = -\bar{\Omega}_y$ and $d\mathbf{x}/dt = \mathbf{c}_g$.

Steady-state view of limits on propagation. One can obtain an intuitive understanding of the fate of Rossby rays directly from the dispersion relation, and ignoring, for now, the time-dependent equations for the meridional wavenumber (Hoskins and Karoly, 1981, after solution 5.23). ENSO events last months, which implies an effectively stationary forcing on the atmosphere, and therefore we expect the wave response to be stationary as well. Start therefore from the dispersion relation for stationary waves, $0 = \omega = u_M k - \beta_M k/(k^2 + l^2)$, and define $K_s = (\beta_M/u_M)^{1/2} = k^2 + l^2$. Based on these only, we can analyze the trapping of the ray by the jet: if $k > K_s$ then l must be imaginary. Suppose the wave that initally satisfies $k < K_s$ propagates poleward from the low latitudes with a constant k (it is conserved because $\partial\Omega/\partial x = 0$), while K_s gets smaller due to latitudinal changes in mean flow and the effective beta (Fig. 13a,b). This implies evanescent behavior in latitudes past the critical latitude at which $k = K_s$, and therefore a trapping of the ray equatorward of the critical latitude.

Allowing for dl/dt. We next consider why the ray is reflected rather than being trapped at the critical latitude as deduced from the above heuristic steady-state argument. In short, this is due to the prognostic dl/dt equation allowing $dl/dt < 0$ when $l = 0$, leading to the ray turning back south (thanks to Jeff Shaman).

The stationary argument based on K_s suggests that the ray has $l = 0$ where $K_s = k$, which only means that northward propagation is halted, not that it turns around. To see how this works: Consider for simplicity solid body rotation, note $K_s = \sqrt{\beta_M/u_M}$, where $u_M = U/\cos(\phi)$, that is in the Mercator projection u_M is a constant, and $\beta_M = 2\Omega\cos^2(\phi)/a$. So, K_s is a decreasing function of latitude (5.25, HK81). See Ray-Tracing-Rossby-Waves-Jeff-solid-test.jpg for an example of ray tracing with solid body rotation, $k = 5$.

The equations are:

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\begin{aligned}
K_s &= \sqrt{k^2 + l^2} \\
\frac{dk}{dt} &= -k\frac{du_M}{dx} - l\frac{dv_M}{dx} + \left(\frac{d^2\overline{Q}}{dxdy}k - \frac{d^2\overline{Q}}{dx^2}l\right)/K_s^2 \\
\frac{dl}{dt} &= -k\frac{du_M}{dy} - l\frac{dv_M}{dy} + \left(\frac{d^2\overline{Q}}{dy^2}k - \frac{d^2\overline{Q}}{dxdy}l\right)/K_s^2 \\
\frac{dx}{dt} &= u_g = u_M + \left((k^2 - l^2)\frac{d\overline{Q}}{dy} - 2kl\frac{d\overline{Q}}{dx}\right)/K_s^4 \\
\frac{dy}{dt} &= v_g = v_M + \left(2kl\ \frac{d\overline{Q}}{dy} - (k^2 - l^2)\frac{d\overline{Q}}{dx}\right)/K_s^4
\end{aligned}
$$

$d\overline{Q}/dy = \beta_M$; for the assumed solid body rotation atmosphere, u_M=constant and v_M=0; Therefore all the derivative terms are zero, but the $d\overline{Q}/dy$ and $d^2\overline{Q}/dy^2$ (due to β);

So,

$$\begin{aligned}
\frac{dk}{dt} &= 0 \\
\frac{dl}{dt} &= \frac{d^2\overline{Q}}{dy^2} k/K_s^2 \\
\frac{dx}{dt} &= u_M + (k^2 - l^2)\frac{d\overline{Q}}{dy}/K_s^4 \\
\frac{dy}{dt} &= 2kl\frac{d\overline{Q}}{dy}/K_s^4
\end{aligned}$$

The key is the second equation. In the northern hemisphere, the initial meridional wavenumber keeps dropping, $dl/dt < 0$, even when $l = 0$. Thus even if l reaches zero, $dy/dt = 0$ by the 4th equation, the ray should not be stuck at that latitude, as the third equation dictates that l decreases. So at the next time step, $dy/dt < 0$. The opposite occurs for the southern hemisphere where $dl/dt > 0$, which eventually sends the ray back northward.

WKB analysis of the wave amplitude during reflection. (Time permitting:) To get some idea of the amplitude of the propagating waves, assume the meridional wavenumber varies slowly in latitude, $l = l(\epsilon y)$, corresponding to a medium that varies slowly, on a scale longer than the wavelength. Use the WKB solution (Bender and Orszag (1978) section 10.1): start with equation 5.18. Substituting into 5.9 this leads to $d^2P/dy^2 + l^2(\epsilon y)P = 0$ for $l^2(\epsilon y)$ defined in 5.20; to transform to standard WKB form, define $Y = \epsilon y$ so that $\epsilon^2 d^2P/dY^2 + l^2(Y)P = 0$; try a WKB solution corresponding to a wave-like exponential with a rapidly varying phase plus a slower correction $P = \exp(S_0(Y)/\delta + S_1(Y))$ to find

$$\epsilon^2[(S_0'/\delta + S_1')^2 + (S_0''/\delta + S_1'')]P + l^2 P = 0;$$

let $\delta = \epsilon$ and then $O(1)$ equation is ${S_0'}^2 + l^2(Y) = 0$ so that $S_0 = i\int l(Y)\,dY$ (if l^2 is nearly constant, this simply reduces to the usual wave solution e^{ily}). Next, consider $O(\epsilon)$ equation which, after using the $O(1)$ equation, is $2S_0'S_1' + S_0'' = 0$ and the solution is $S_1' = -S_0''/(2S_0') = -(dl/dY)/(2l) = -d/dY(\ln l^{1/2})$ so that $S_1 = \ln l^{-1/2}$ which means that the wave amplitude is $l^{-1/2}$. This gives the solution in Hoskins and Karoly (1981) equation (5.21, 5.23), see further discussion there.

Constant angular momentum flow. We can calculate analytically Rossby rays for a constant angular momentum flow, $u_M \equiv U/\cos\phi = \bar{\omega}a$ for some constant $\bar{\omega}$, and with $\beta_M = (2\cos^2\phi)(\Omega + \bar{\omega})/a$ (section 5c, page 1192, Fig. 12) and then the one using realistic zonal flows (Figs. 13, 14, 15, etc);

Nonlinear effects and baroclinic waves. Finally, later works showed that stationary linear barotropic Rossby waves excite nonlinear eddy effects which may eventually dominate the teleconnection effects.

For baroclinic atmospheric waves,

$$0 = \omega = \bar{\Omega}(k, l, y) = u_M k - \frac{\beta_M k}{k^2 + l^2 + L_R^{-2}},$$

so that $K_s^2 = (\beta_M / u_M) - L_R^{-2} = k^2 + l^2$. The smaller K_s implies that such baroclinic waves are more easily trapped (that is, trapped for a smaller value of k than barotropic waves). They are typically trapped near the equator within a scale of the Rossby radius of deformation L_R which is some 1000 km or so (see discussion on page 1195 left column).

References

C. M. Bender and S. A. Orszag. *advanced mathematical methods for scientists and engineers.* McGraw-Hill, 1978.

B J Hoskins and D J Karoly. The steady linear response of a spherical atmosphere to thermal and orographic forcing. *Journal of the Atmospheric Sciences*, 38(6):1179–1196, 1981. doi: 10.1175/1520-0469(1981)038⟨1179:TSLROA⟩2.0.CO;2.