Notes on
Ice Dynamics Preliminaries
for
Climate Dynamics (EPS 231)

Eli Tziperman

August 7, 2025

Contents

1 Introduction to ice dynamics

Rate of strain definition from Kundu and Cohen (2002) sections 3.6 and 3.7, pages 56–58; stress and deviatoric stress and stress-rate of strain relationship and Glenn's law from section 2.3, pages 13–15 of Van-Der-Veen (1999).

2 Parabolic profile of ice sheets

Plastic ice sheet. Consider first a simple version assuming ice is a plastic material (WH notes p 101). Consider a balance of forces for a glacier that is symmetric in longitude x. The glacier height as a function of latitude is $h(y)$. The balance of forces on a slice of the glacier between latitudes $(y, y+dy)$ is between hydrostatic pressure integrated along the face of the slice, and stress applied by bottom friction

$$\int_0^{h(y+dy)} \rho_{ice} g z dz - \int_0^{h(y)} \rho_{ice} g z dz = \tau(y, z=0) dy$$

or simply

$$h(y)\frac{dh}{dy}\rho_{ice}g=\tau(y,z=0)=\tau_0$$

where we assume that the bottom is at the yield stress τ_0 (i.e., glacier in a "critical" state). In other words, we assume perfect plasticity: the glacier yields to the hydrostatic-induced stress at the above critical stress. The solution to the last equation gives the desired parabolic profile that is not a bad fit to observations,

$$\frac{1}{2}h(y)^2=\frac{\tau_0}{g\rho_{ice}}(y-y_0).$$

Viscous ice sheet. Now the more accurate expression shown by the solid line in Fig 11.4 in Paterson (1994): the following derivation roughly follows Chapter 5, p 243 Equations 6–10 and p 251, Equations 18–22 from Paterson (1994),

$$\dot{\epsilon}_{xz}=\frac{1}{2}\frac{du}{dz}=A\tau_{xz}^n=A\left(\rho g(h-z)\frac{dh}{dx}\right)^n.$$

Integrate from $z=0$ to z, and use the b.c. $u(z=0)=u_b$,

$$u(z)-u_b=-2A\left(\rho g\frac{dh}{dx}\right)^n\frac{(h-z)^{n+1}}{n+1}+2A\left(\rho g\frac{dh}{dx}\right)^n\frac{h^{n+1}}{n+1}.$$

Let $u_b=0$ (no sliding) and average the velocity in z,

$$\begin{aligned}\bar{u} &= (1/h)\int_0^h dz2A\left(\rho g\frac{dh}{dx}\right)^n\frac{(h-z)^{n+1}}{n+1}-2A\left(\rho g\frac{dh}{dx}\right)^n\left(\frac{h^{n+1}}{n+1}\right)\\ &= \frac{2A}{(n+1)}\left(\rho gh\frac{dh}{dx}\right)^n h\left(\frac{1}{n+2}-1\right)\\ &= \frac{2A}{(n+2)}\left(-\rho gh\frac{dh}{dx}\right)^n h. \qquad (1)\end{aligned}$$

Next, we use continuity, assuming a constant accumulation of ice at the surface, $d(h\bar{u})/dx=c$ which implies together with the last equation

$$cx+K_1=h\bar{u}=\frac{2A}{(n+2)}\left(-\rho gh\frac{dh}{dx}\right)^n h^2=K_2\left(h\frac{dh}{dx}\right)^n h^2,$$

where ablation is assumed to occur only at the edge of the ice sheet at $x=L$. The last equation may be written as

$$(K_3x+K_4)^{1/n}dx=h^{2/n+1}dh$$

and solved using boundary conditions of $h(x=0)=H$ and $h(x=L)=0$ to obtain

$$(x/L)^{1+1/n} + (h/H)^{2/n+2} = 1.$$

This last equation provides a better fit to obs as shown in Paterson Fig 11.4 (also shown in WH notes).

3 Shallow ice approximation

Let (s, b) be the ice surface and ice bottom heights. Then, the momentum equations, Glenn's law, the top and bottom boundary condition, and mass conservation equations are,

$$\begin{aligned} 0 &= -p_x + \frac{\partial \tau_{xz}}{\partial z}, \quad p_z = -\rho g, \\ u_z &= 2A|\tau_{xz}|^{n-1}\tau_{xz} \\ u(z=b) = 0, &\quad \tau_{xz}(s) = 0, \quad p(s) = 0 \\ s_t + \partial_x q = a, &\quad \text{where} \quad q = \int_b^s u\,dz \end{aligned}$$

Note that in the x-momentum equation, the time rate of change, nonlinear advection terms and the Coriolis term are all negligible. The stress term on the RHS depends on the velocity via the dependence of the stress on the range of strain. Integrating the hydrostatic equation from the surface, one finds

$$p(x,z) = (s(x) - z)\rho g.$$

Substitute in the momentum equation to find,

$$\frac{\partial \tau_{xz}}{\partial z} = \rho g s_x$$

Integrate using the zero stress condition at the top, $z = s$,

$$\tau_{xz} = \rho g (z - s) s_x$$

substitute in Glenn's law,

$$u_z = 2A(\rho g)^n (z-s)^n |s_x|^{n-1} s_x.$$

Integrate from the bottom to z to find u, using b.c. of zero velocity at the bottom,

$$u(z) = \frac{2A}{n+1}(\rho g)^n [(s-z)^{n+1} - (s-b)^{n+1}]|s_x|^{n-1} s_x,$$

and then again from bottom to top to find q,

$$\begin{aligned} q &= \frac{2A}{n+1}(\rho g)^n [\frac{1}{n+2}(s-b)^{n+2} - (s-b)^{n+2}]|s_x|^{n-1}s_x \\ &= -\frac{2A}{(n+2)}(\rho g)^n (s-b)^{n+2}|s_x|^{n-1}s_x. \end{aligned}$$

The mass conservation equation can therefore be written as a nonlinear diffusion equation for the ice surface elevation, s, with a diffusion coefficient D that depends on s,

$$s_t - \partial_x(Ds_x) = a,$$
$$D = \frac{2A}{(n+2)}(\rho g)^n (s-b)^{n+2}|s_x|^{n-1}.$$

Note that the shallow ice approximation and the shallow water approximation are making very different assumptions and reflect very different physical balances despite the similar names…

References

Kundu, P. and Cohen, I. M. (2002). *Fluid mechanics*. Academic Press, second edition.

Paterson, W. (1994). *The Physics of Glaciers*. Pergamon, 3rd edition.

Van-Der-Veen, C. (1999). *Fundamentals of Glacier Dynamics*. A.A. Balkema, Rotterdam, The Netherlands.