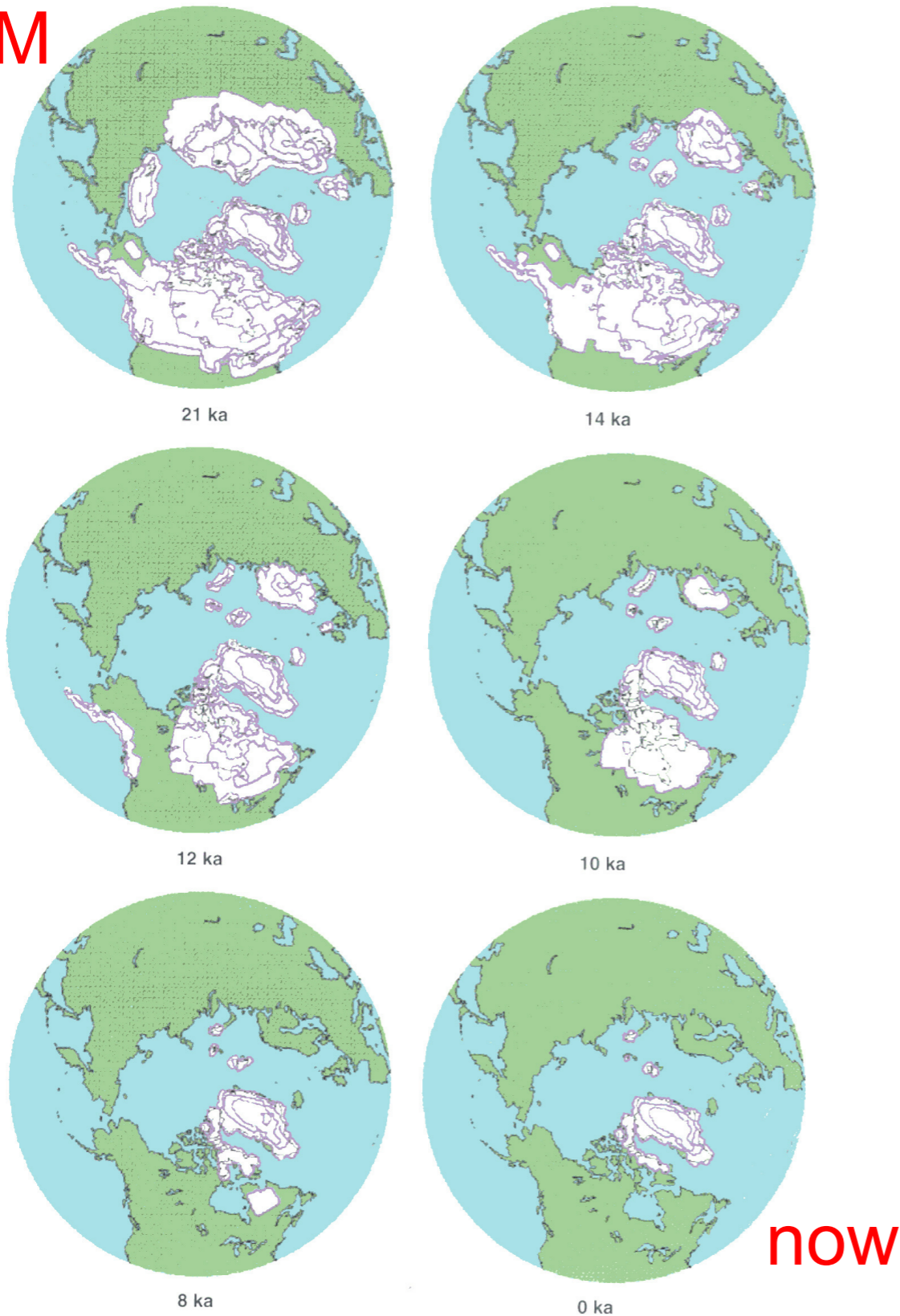


Glacial cycles

EPS 231 Climate dynamics
Eli Tziperman

Observed glacial cycle characteristics

LGM

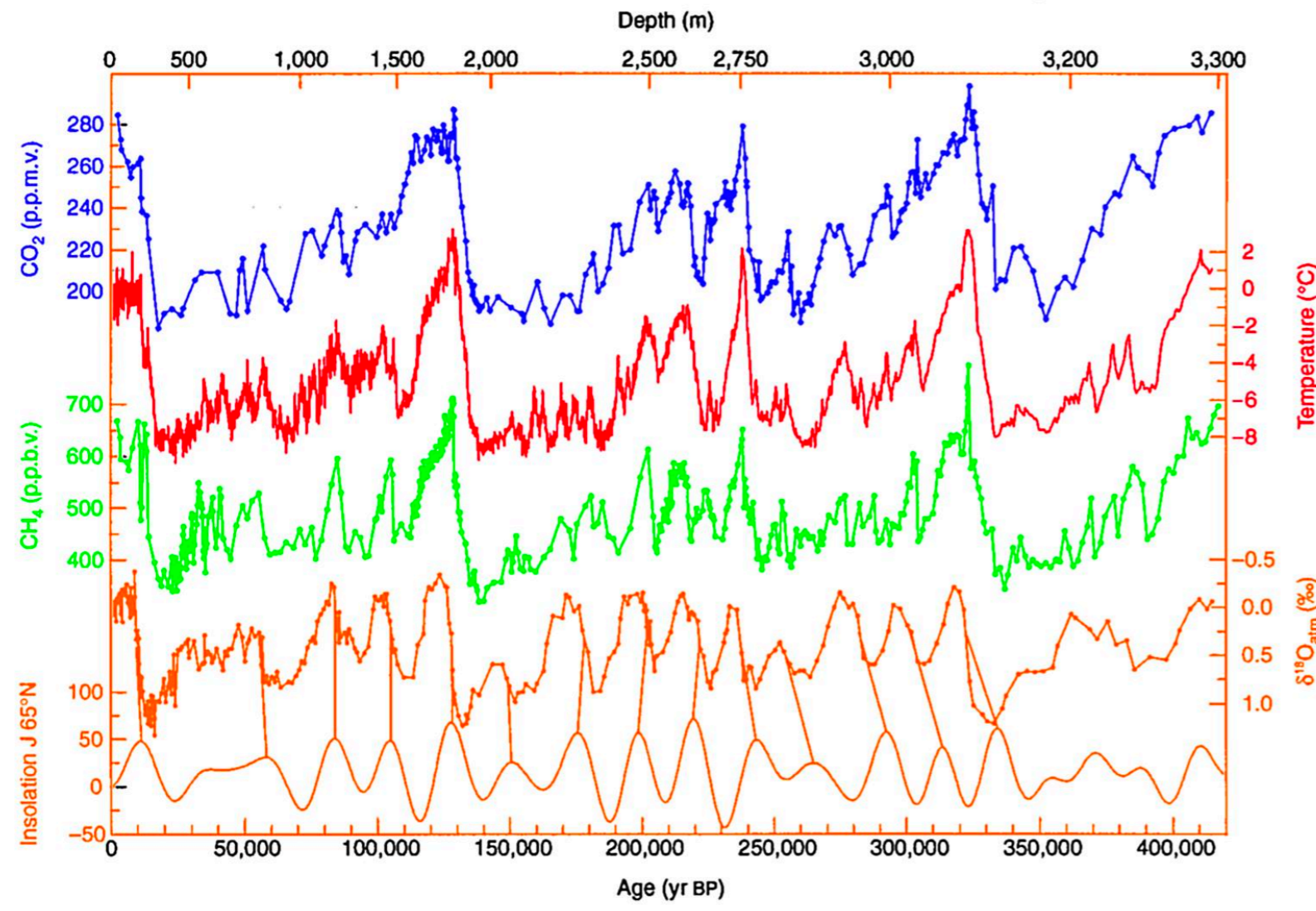


Last Glacial Maximum: 21 kyr
ice sheet elevation 2–3 km
sea level lower by 130 m

now

Fig. 4. Thickness isopachs for the ICE-4G model for a sequence of times beginning at Last Glacial Maximum at 21 ka and ending at the present. The contour interval is 1 km.

Observed glacial cycle characteristics



Ice core climate proxy record, Vostok (78S 106E), Antarctica; depth of drilling: 3623 meter; age range: 420,000 years

https://en.wikipedia.org/wiki/Vostok_Station

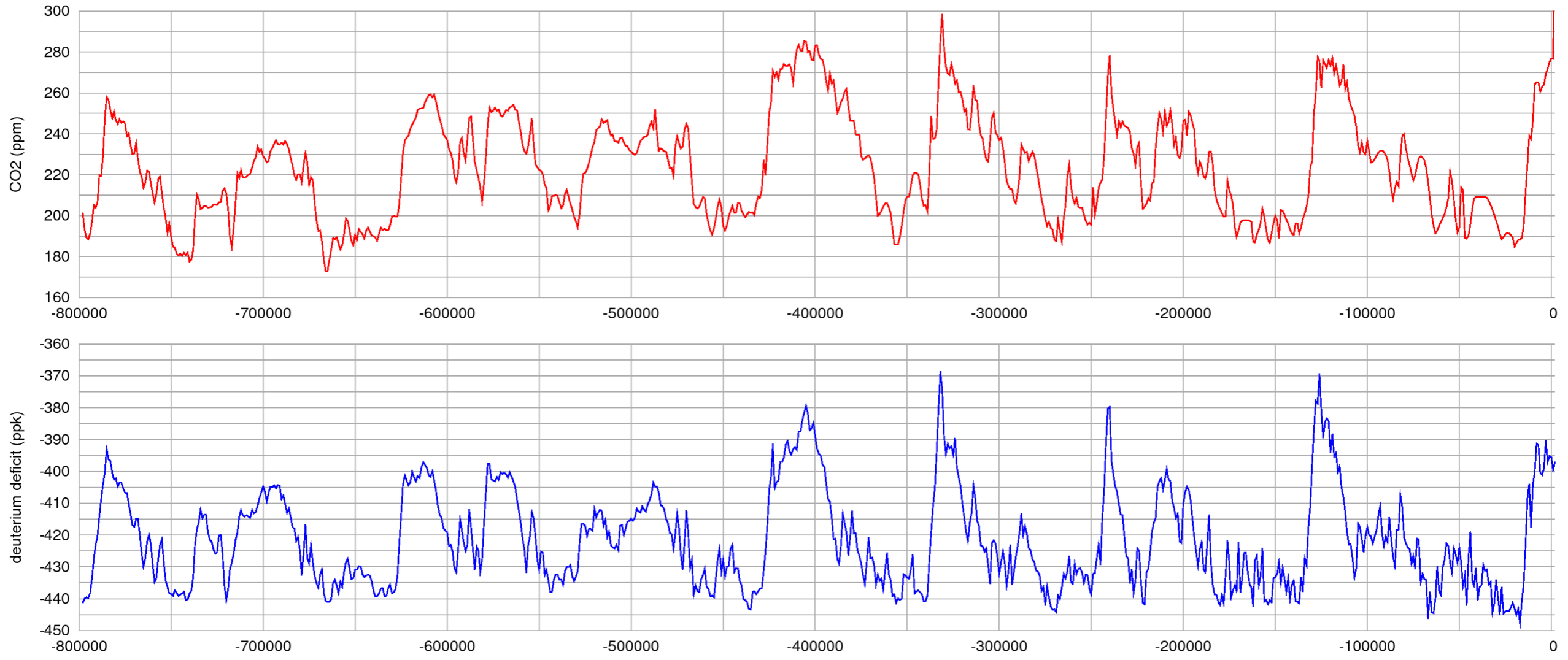
$$\delta^{18}\text{O} = \left(\frac{\left(\frac{^{18}\text{O}}{^{16}\text{O}} \right)_{\text{sample}}}{\left(\frac{^{18}\text{O}}{^{16}\text{O}} \right)_{\text{standard}}} - 1 \right) \times 1000 \text{ ‰}$$



Ice core taken out of drill, Byrd, Antarctica (L. Thompson)

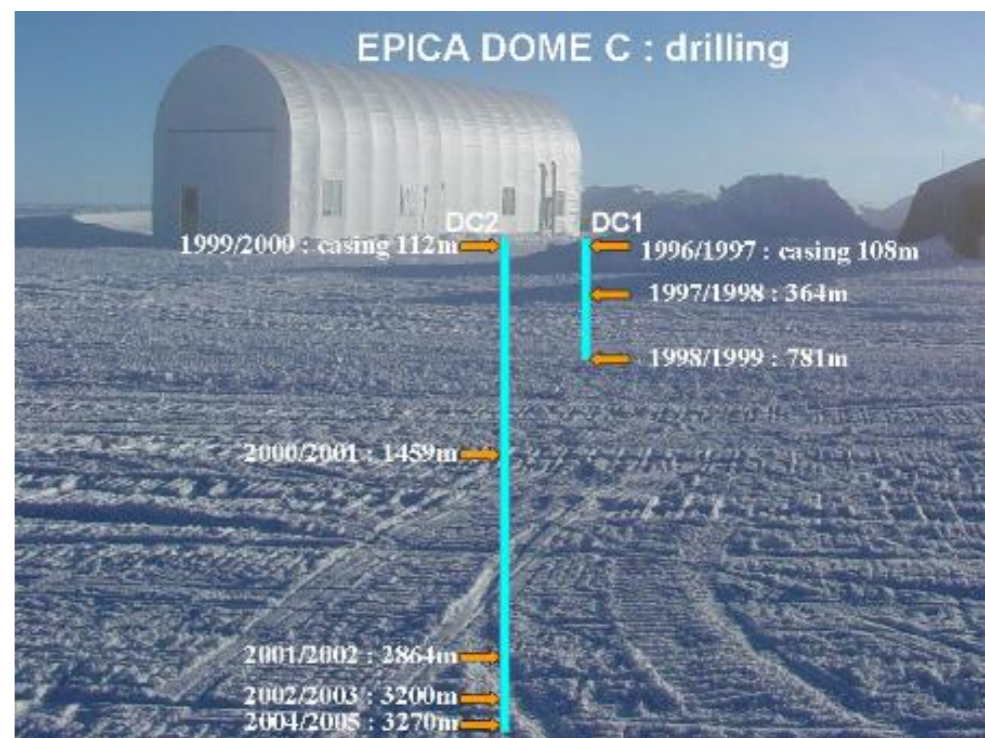
https://en.wikipedia.org/wiki/File:Icecore_4.jpg

Observed glacial cycle characteristics

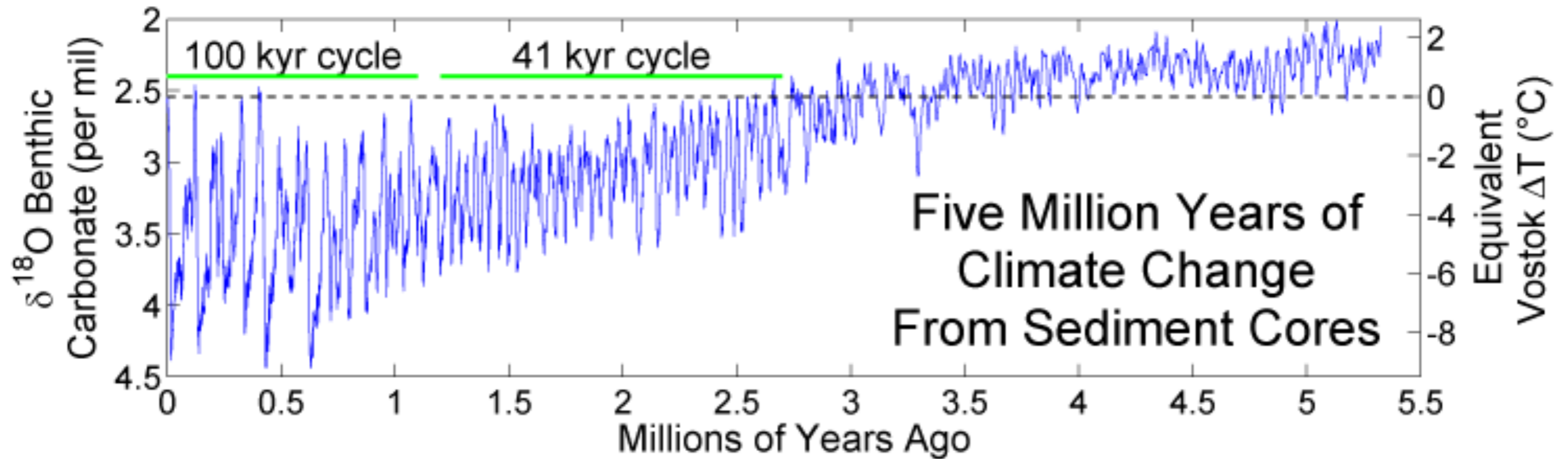


Erica ice core, 800,000 years

Eight glacial cycles from an Antarctic ice core
EPICA community members, 2004



Observed glacial cycle characteristics



Deep sea core, past 5 M yrs

Lisiecki and Raymo (2005) LR04 Benthic Stack

https://en.wikipedia.org/wiki/Marine_isotope_stages

Observed glacial cycle characteristics

A Pliocene-Pleistocene stack of 57 globally distributed benthic $\delta^{18}\text{O}$ records

Lisiecki, Lorraine E., and Maureen E. Raymo. "A Pliocene-Pleistocene stack of 57 globally distributed benthic $\delta^{18}\text{O}$ records." *Paleoceanography* (2005)

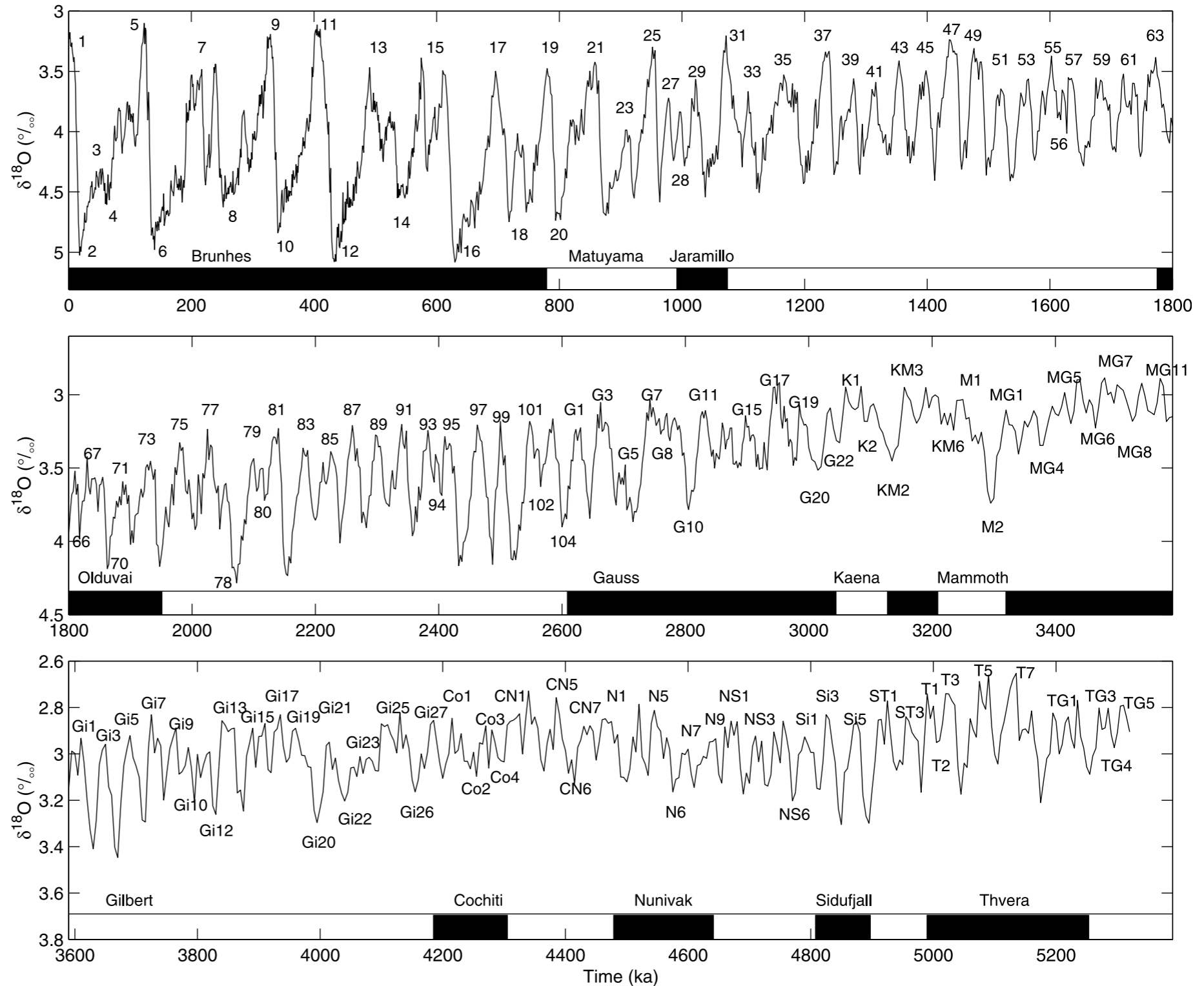
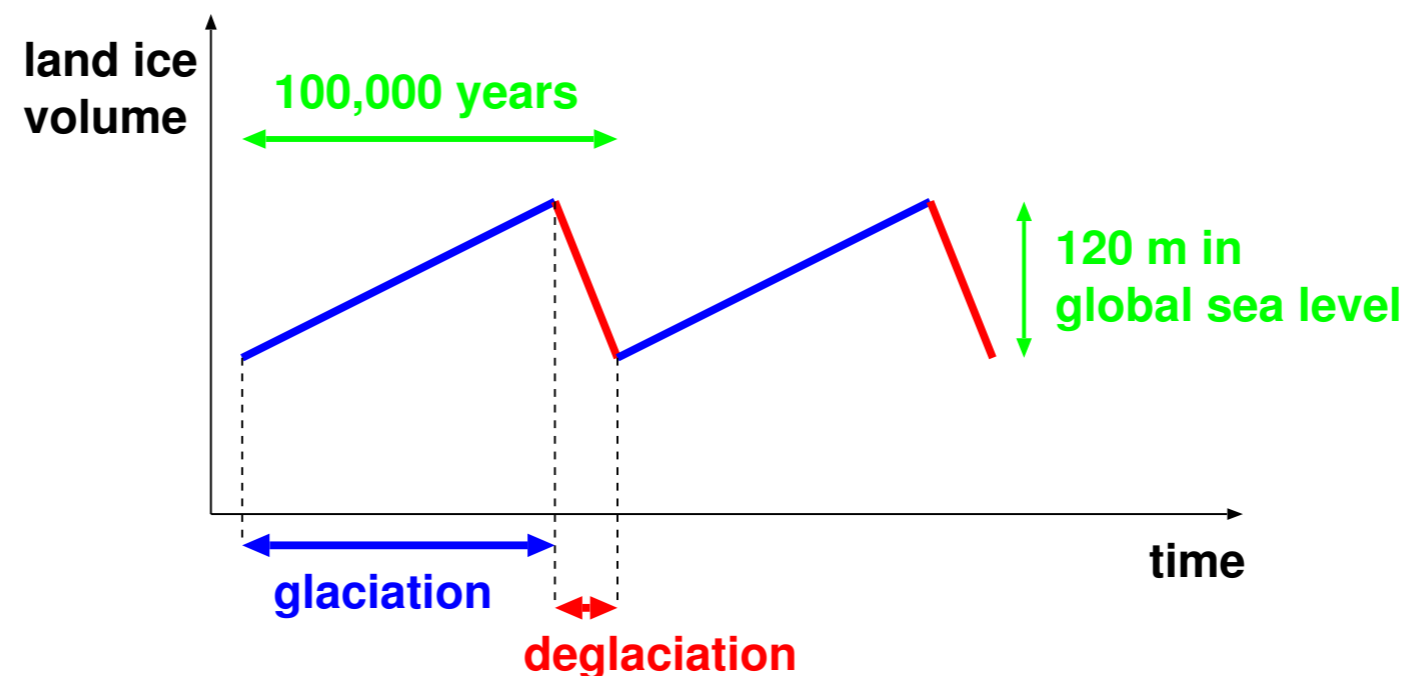


Figure 4. $\delta^{18}\text{O}$ stack constructed by the graphic correlation of 57 globally distributed benthic $\delta^{18}\text{O}$ records. **The scale of the vertical axis changes across panels.**

Characteristics of glacial cycles to be explained by a successful theory...

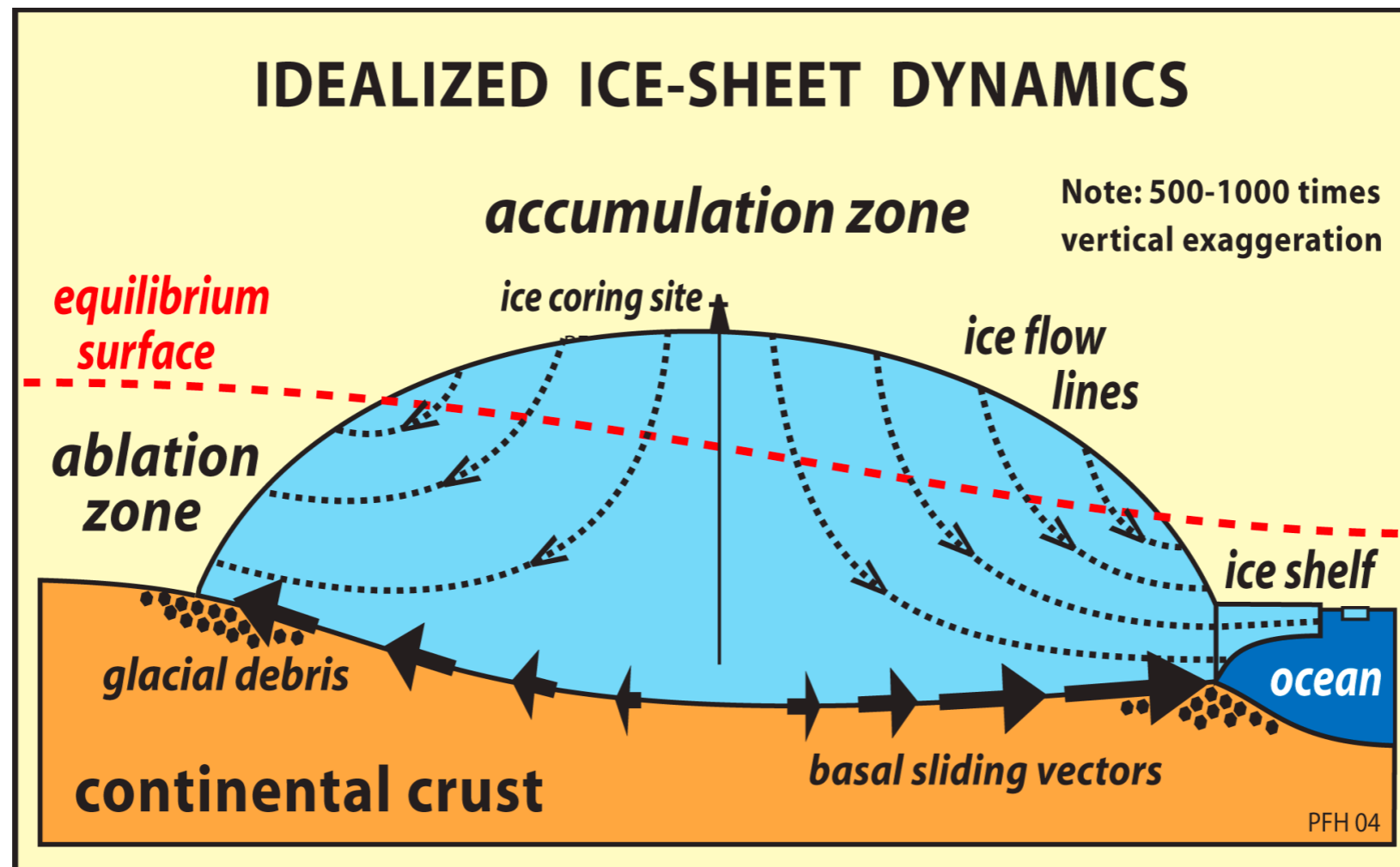
- 100 kyr time scale
- Saw-tooth structure: long glaciations (~90,000 yr), short deglaciations (10,000 yr)
- Transition from 41 kyr to 100 kyr glacial cycles ~800 kyr ago
- Atmospheric CO₂ variations during glacial cycles
- Global scale: both northern & southern hemispheres



Glacial cycle mechanism ingredients

1. energy balance and albedo feedback
2. accumulation, ablation (mass balance) as a function of ice sheet height, equilibrium line
3. Milankovitch forcing
4. ice flow and Glenn's law
5. parabolic ice sheet profile
6. ice streams, calving
7. dust loading and enhanced ablation
8. temperature-precipitation feedback
9. shallow ice approximation
10. isostatic adjustment
11. geothermal heating

equilibrium line, accumulation zone, ablation zone



<http://www.snowballearth.org/slides/Ch10-7.gif>

Milankovitch forcing

Proxy record: 41 kyr and 100 kyr oscillations

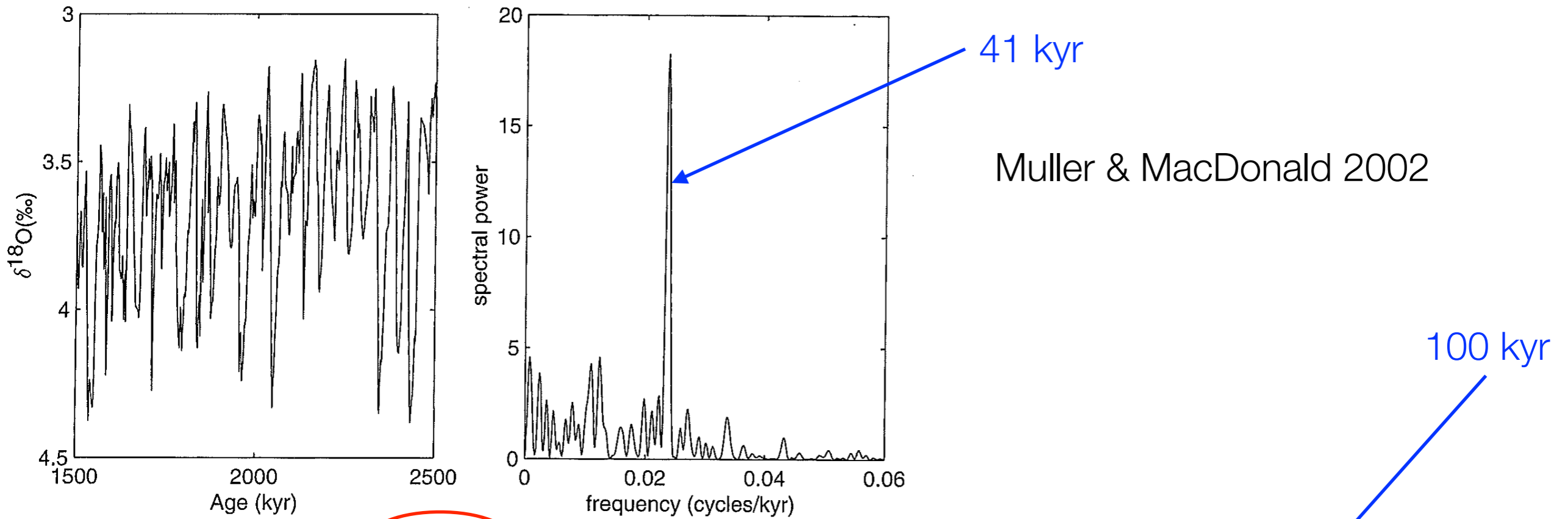


Fig. 2.1. Site 607 from 1.5 to 2.5 kyr: $\delta^{18}\text{O}$ data and spectrum.

The spectral peak of proxy record (say of ice volume) moved from 41 kyr to 100 kyr at about 800 kyr BP

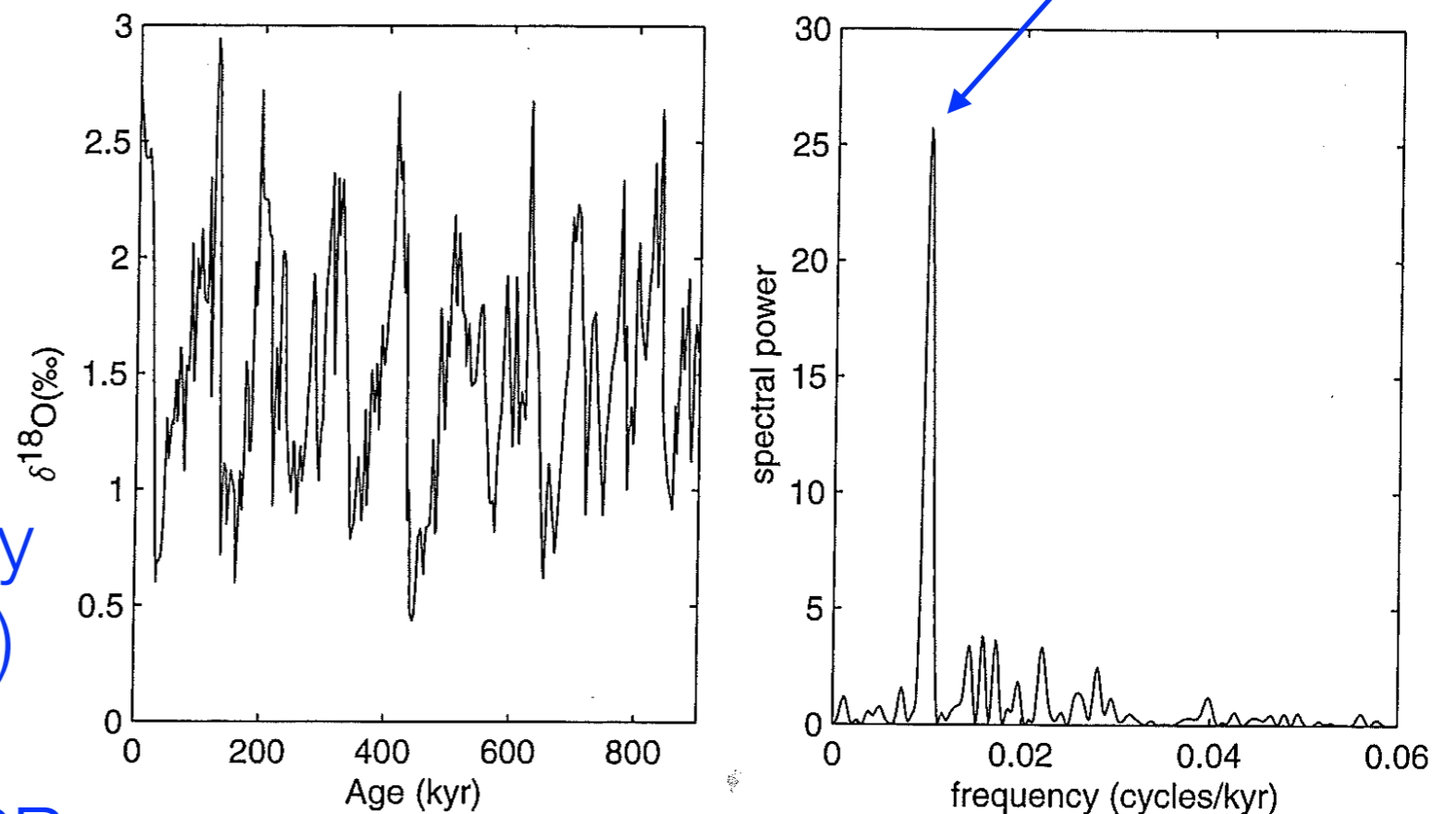
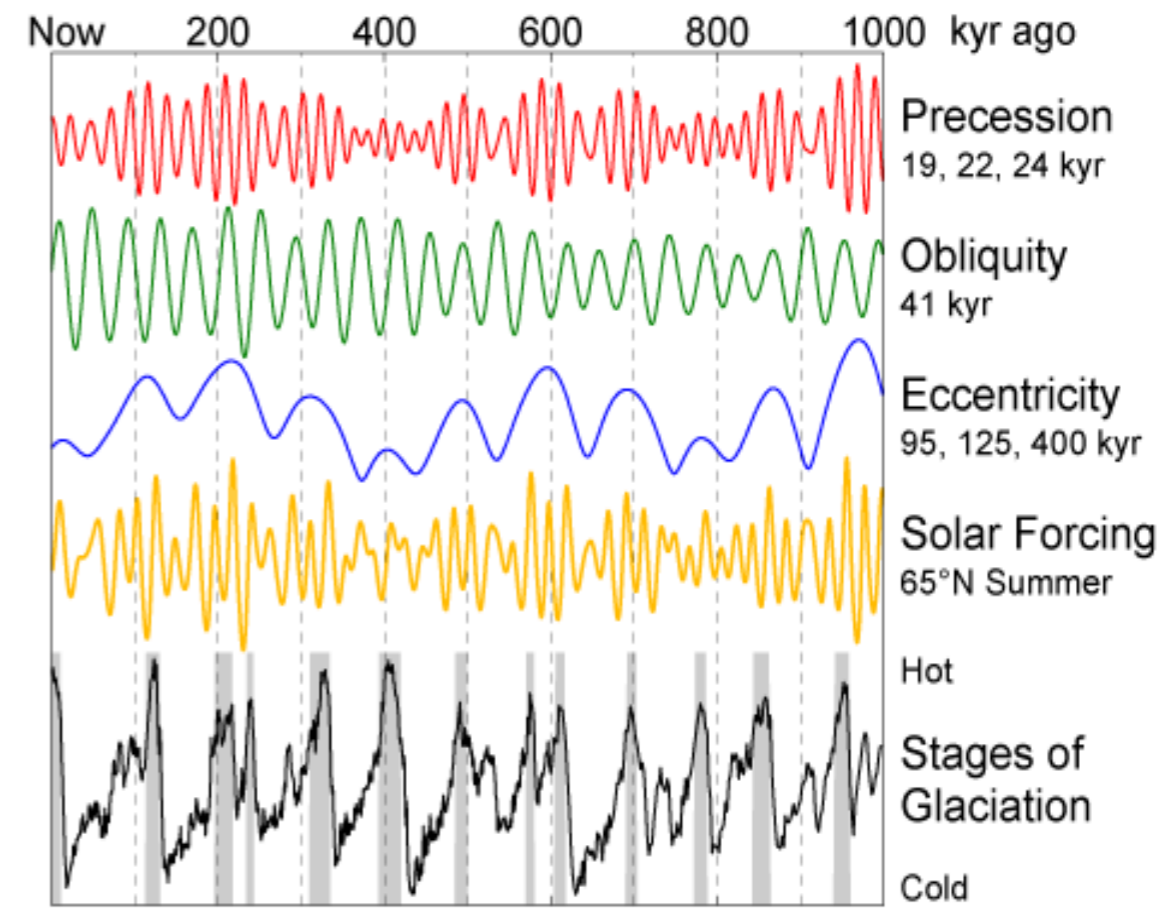
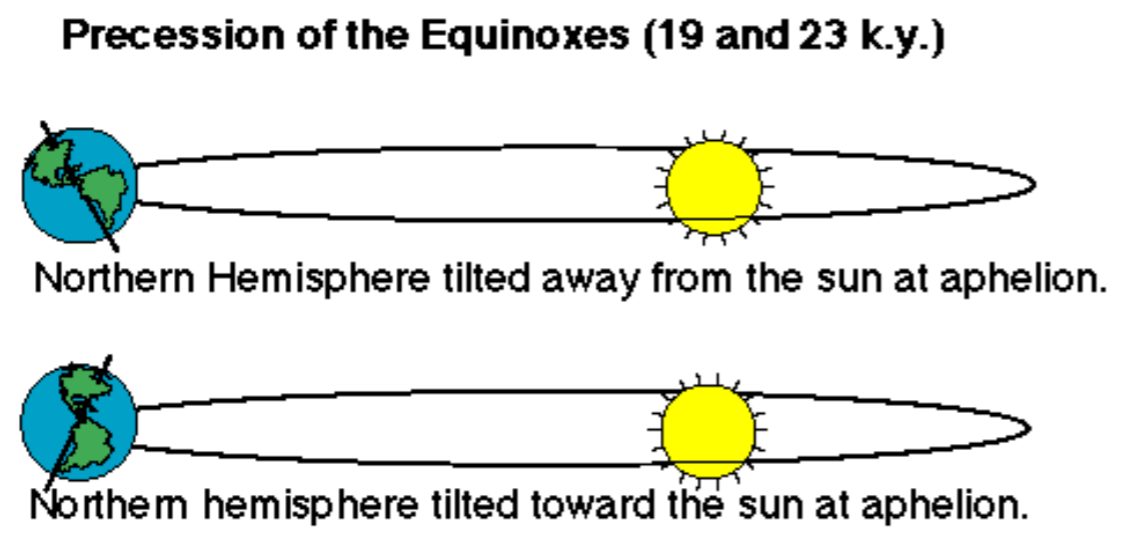
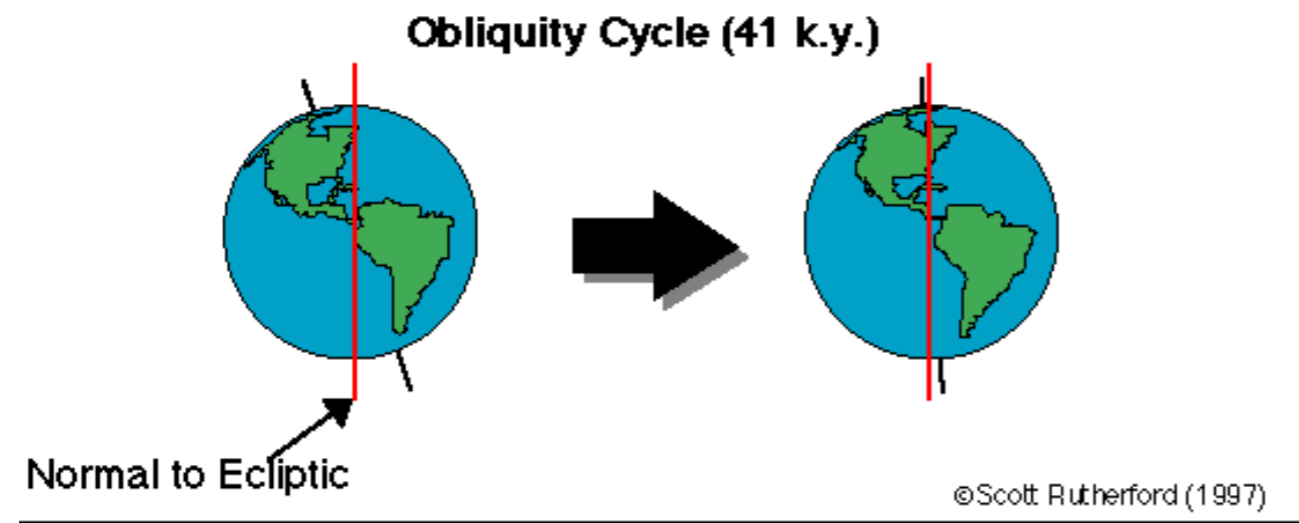
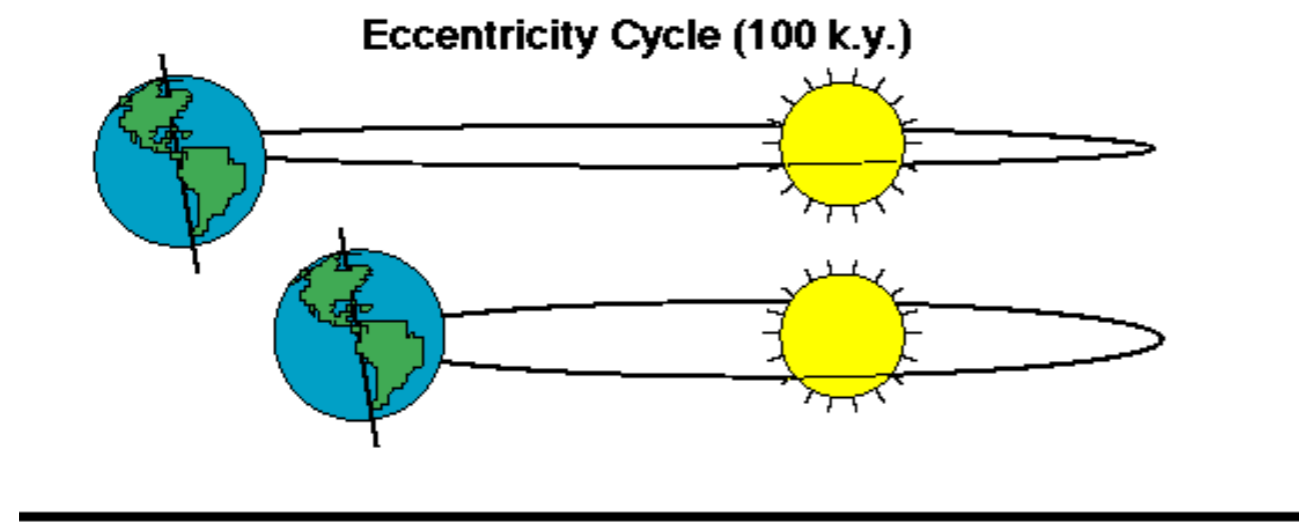


Fig. 2.2. Site 659 $\delta^{18}\text{O}$ and spectrum 0 to 900 kyr.

Milankovitch forcing

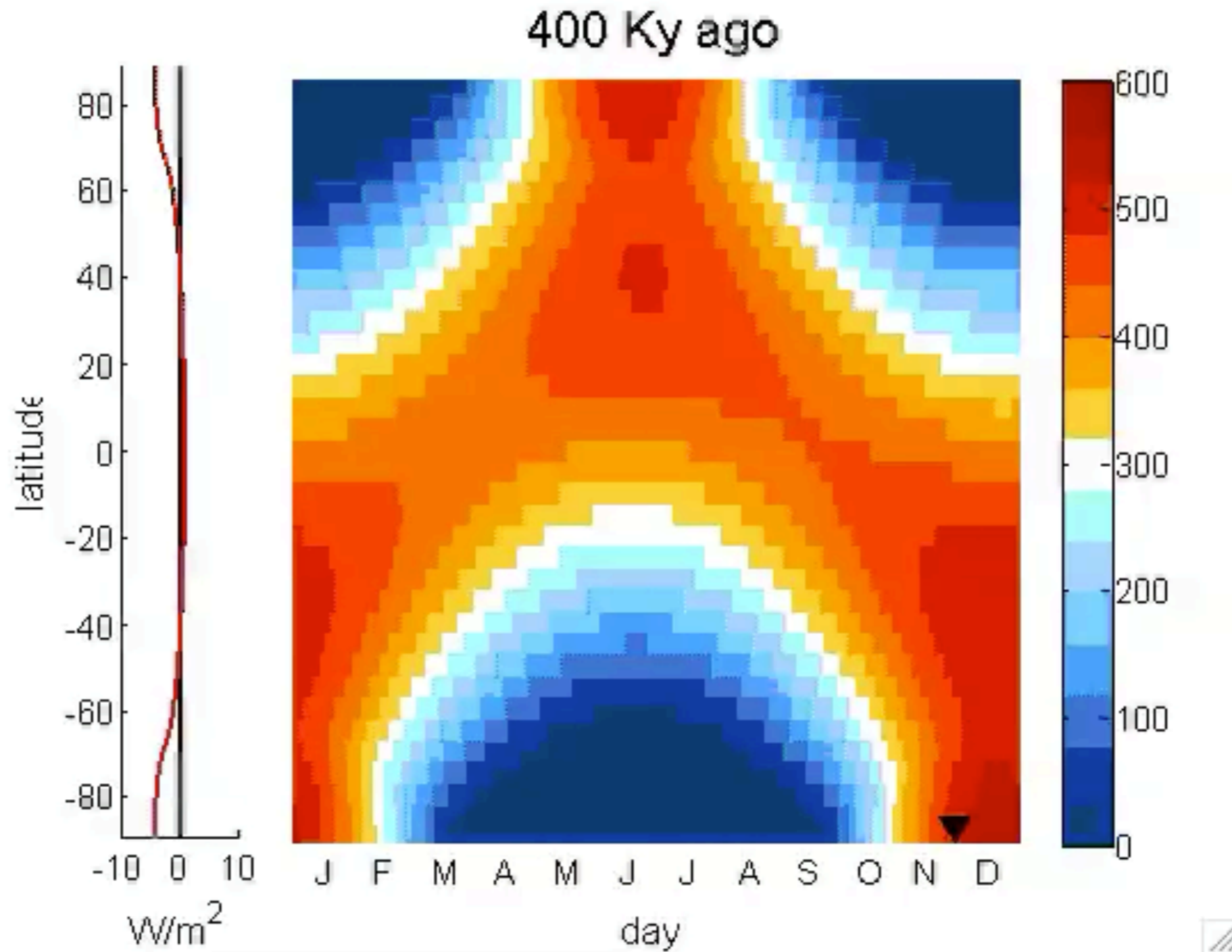


Milankovitch cycles over the past 1,000,000 years. Source: *Global Warming Art*

There seems to be a connection between glaciations & summer insolation at 65N.

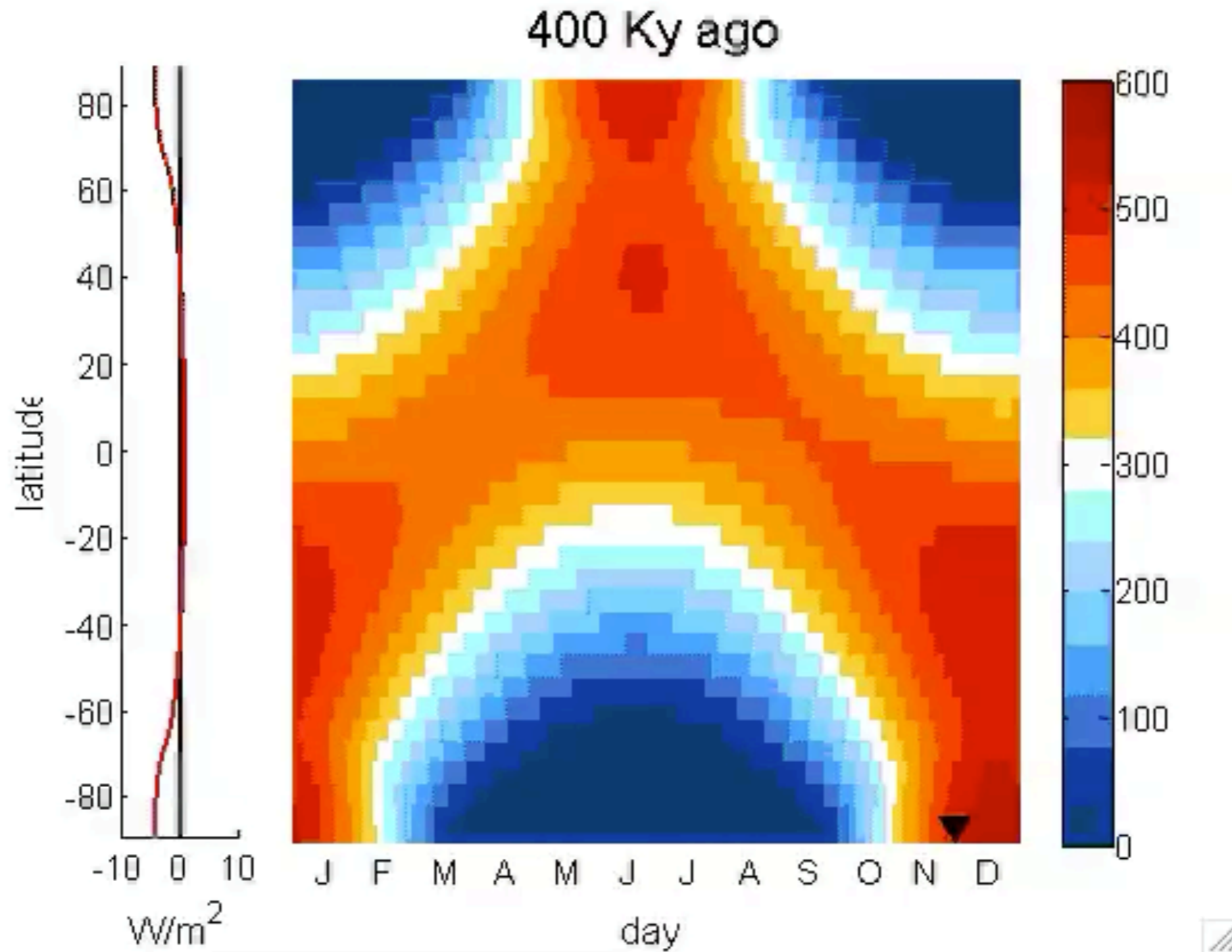
Hays et al 1976: insolation is the “pacemaker” of glacial cycles. How does this work...?

Milankovitch forcing



<https://www.people.fas.harvard.edu/~phuybers/Inso/index.html>

Milankovitch forcing



<https://www.people.fas.harvard.edu/~phuybers/Inso/index.html>

Milankovitch forcing

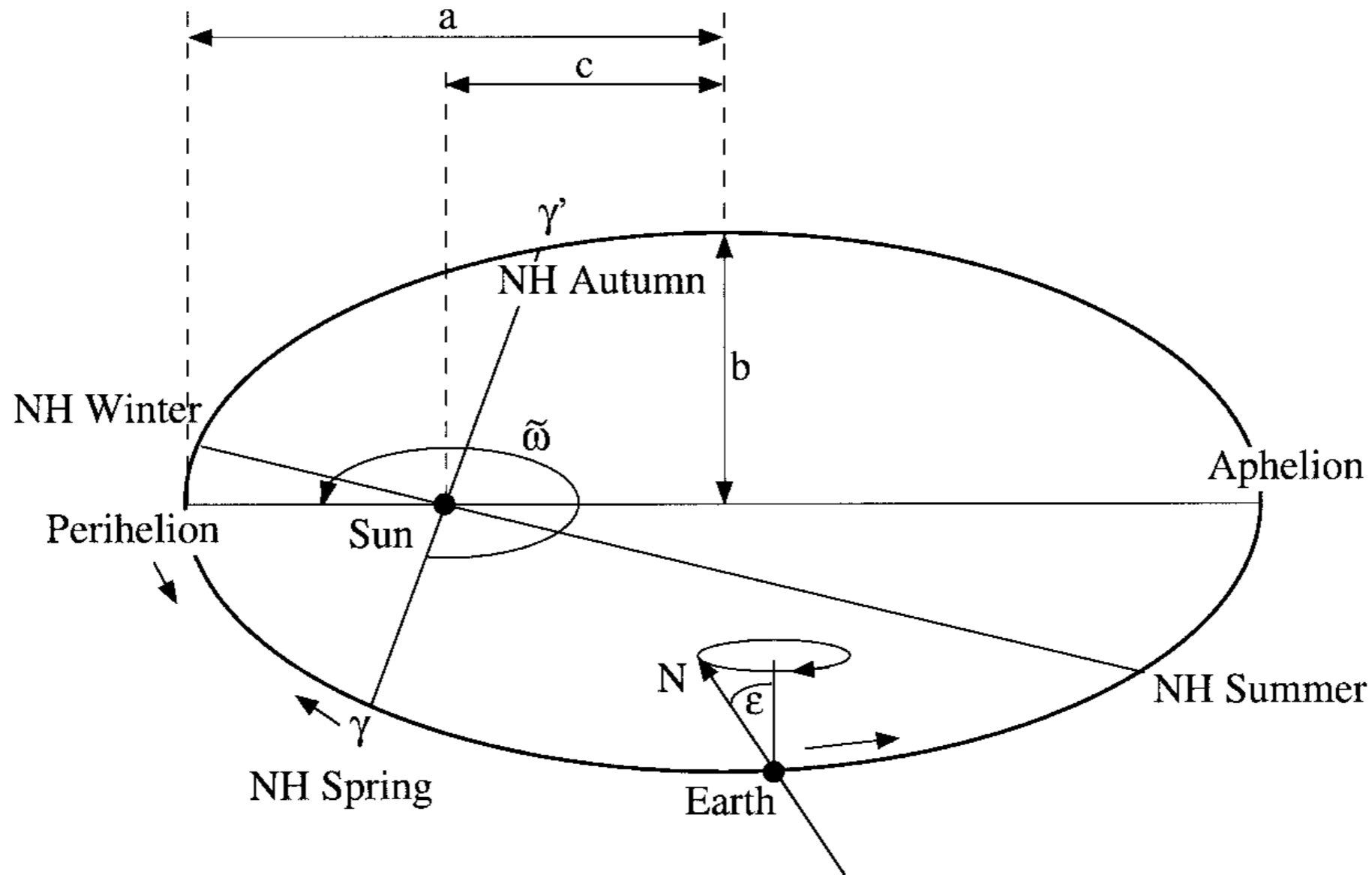


Figure 2. The orbital parameters of the Earth. Eccentricity e is defined as $e = c/a$, where a is the semimajor axis and c is the distance between the focus and the center of the ellipse. The semiminor axis b is then given by Pythagoras's theorem ($a^2 = b^2 + c^2$, which gives $b = a\sqrt{1 - e^2}$). The current eccentricity value is $e = 0.0167$, which means that the Earth's orbit is very close to a circle. The tilt of the Earth's axis with respect to the orbital plane is the obliquity ϵ (current value is $\epsilon = 23.44^\circ$). This tilt implies that the Earth equatorial plane intersects with its orbital plane, the intersection defining the $\gamma\gamma'$ line and the position of equinoxes and solstices. In the current configuration the Earth is closest to the Sun (perihelion) around January 3, just a few weeks after the Northern Hemisphere winter. This position, relative to the vernal equinox γ , is measured by the $\tilde{\omega}$ angle.

current Earth eccentricity=0.016722

Muller & MacDonald 2002

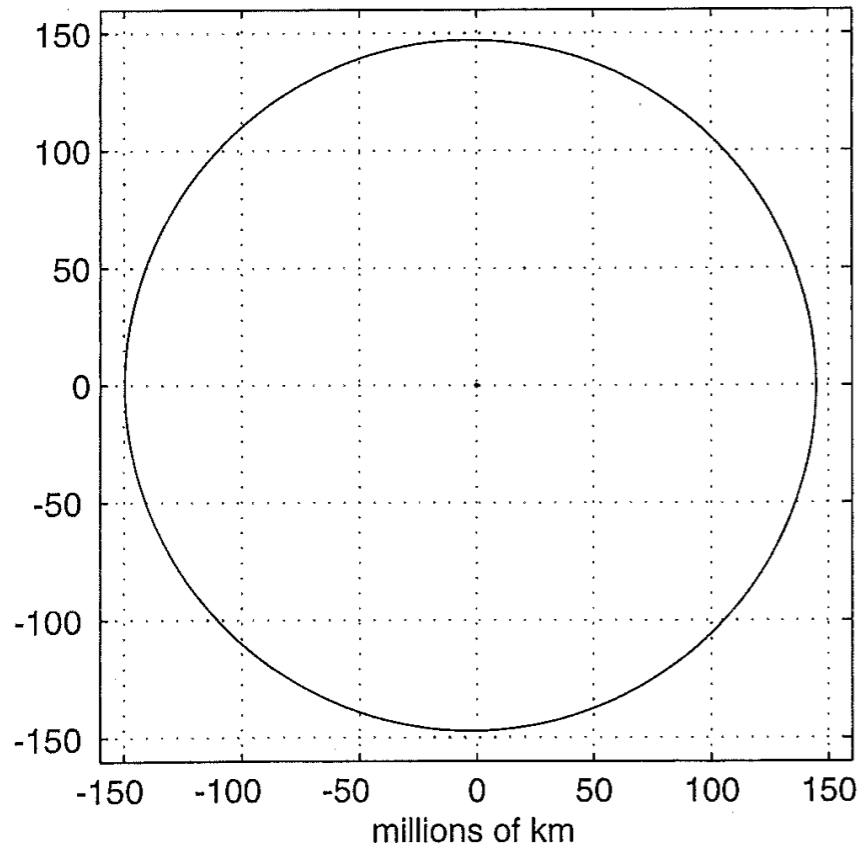


Fig. 2.5. Earth's orbit, to scale.

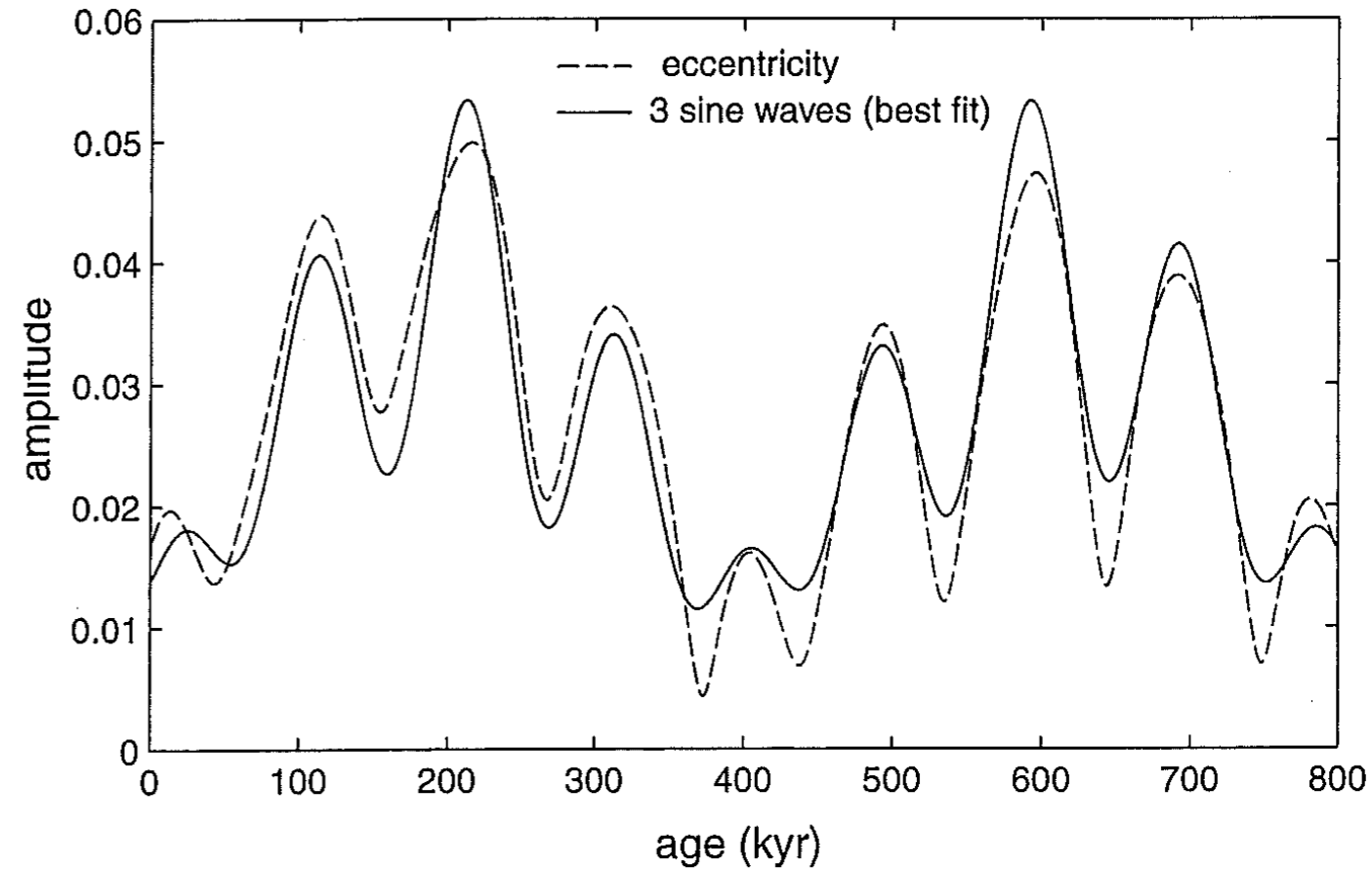


Fig. 2.7. Sum of sine waves with periods 95, 125 and 400 kyr.

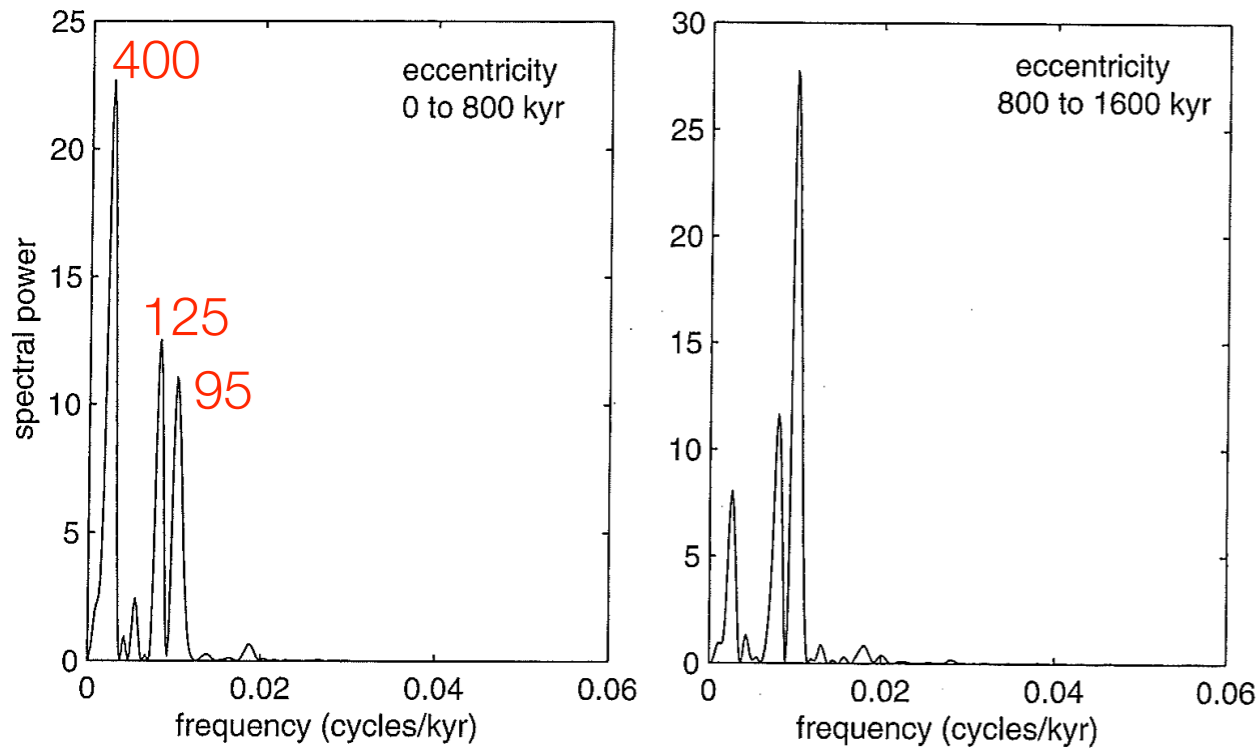
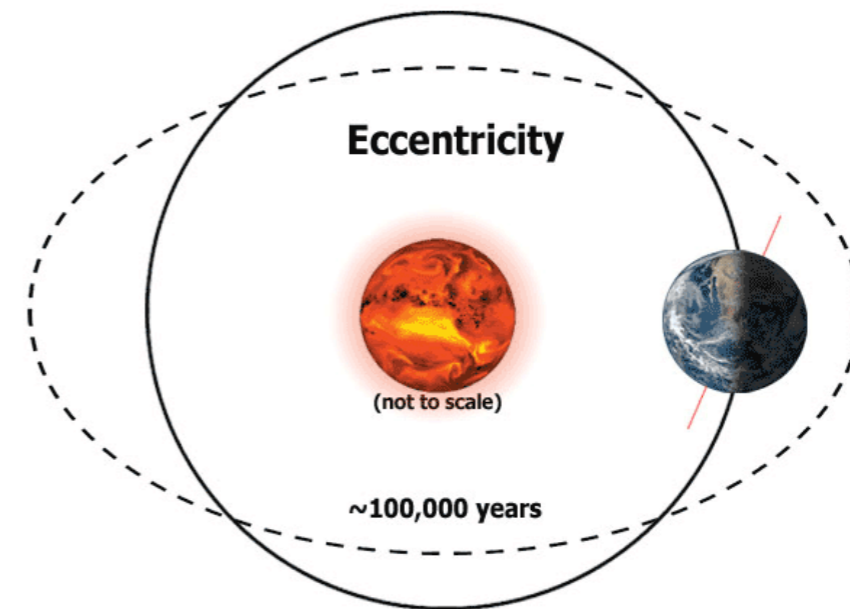
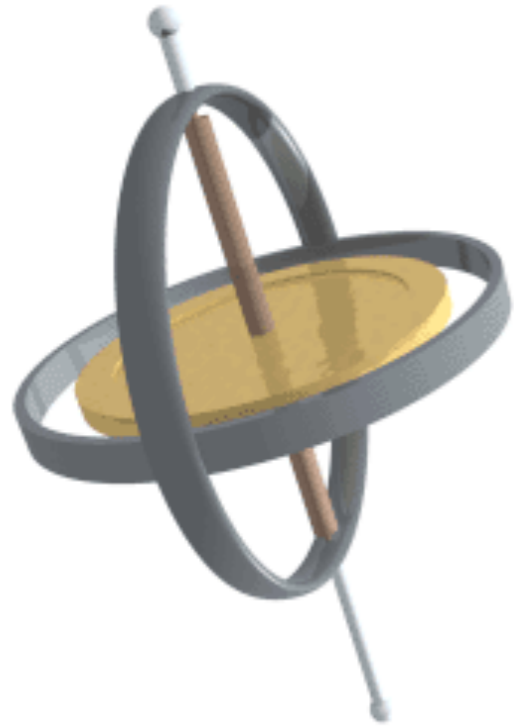


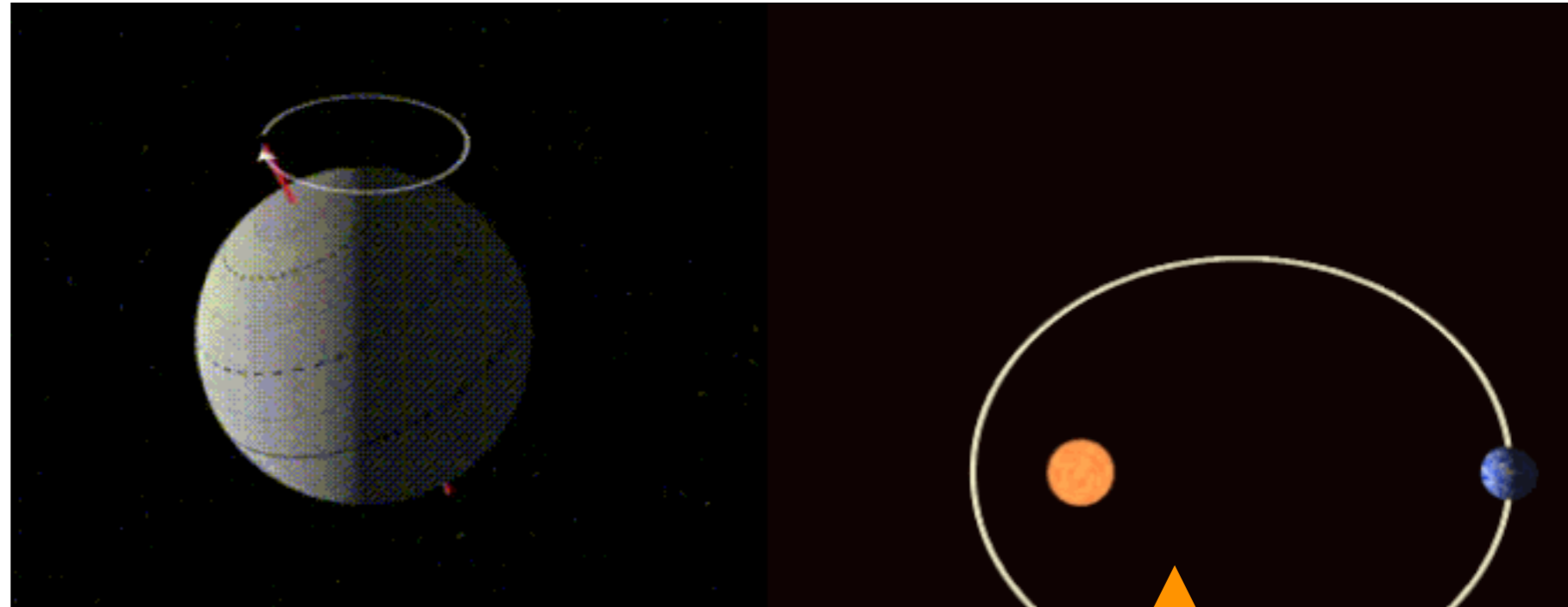
Fig. 2.9. Spectrum of eccentricity.



Milankovitch forcing: precession of orbit and of axis of rotation



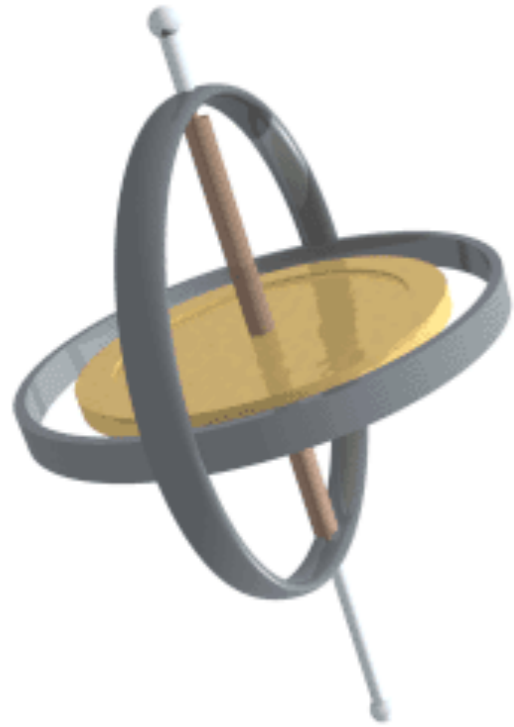
[https://commons.wikimedia.org/wiki/
File:Gyroscope_precession.gif](https://commons.wikimedia.org/wiki/File:Gyroscope_precession.gif)



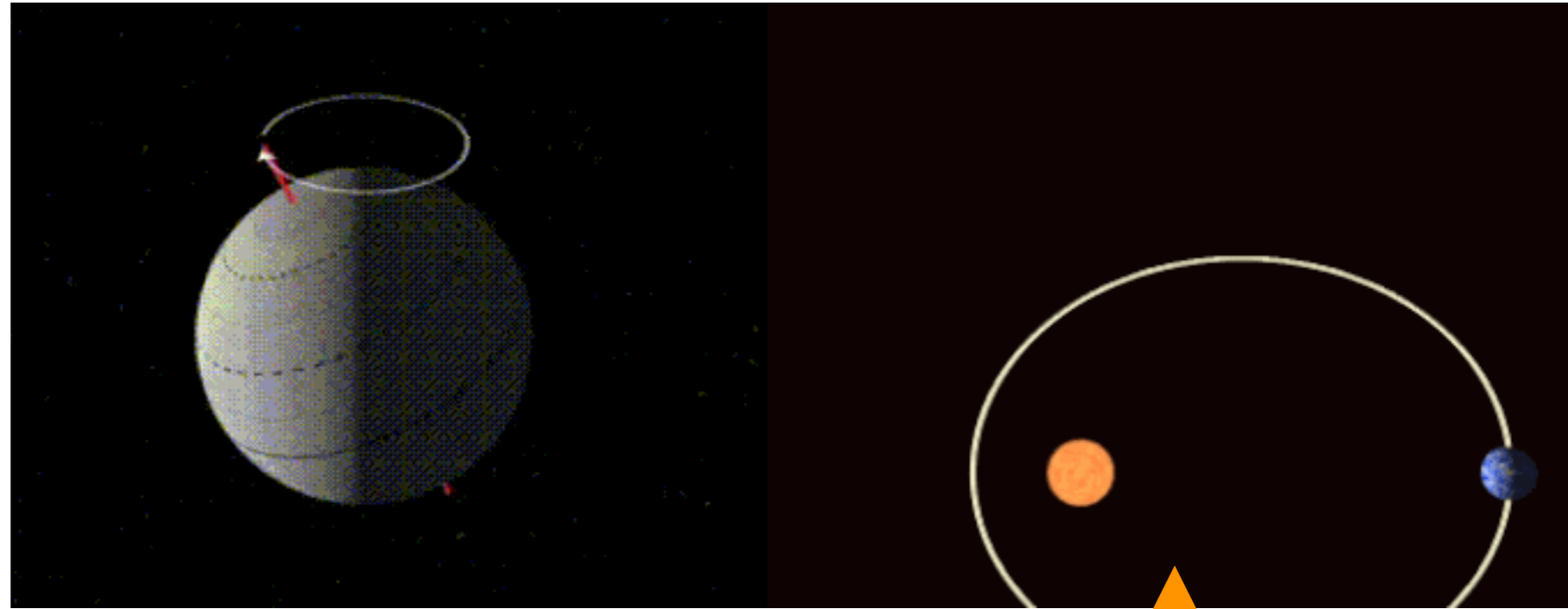
Rotational precession [left] and orbital precession [right]. [Robert Simmon / NASA; WillowW / Wikimedia Commons] downloaded from <https://www.technologyshout.com/there-is-a-new-hypothesis-about-how-uranus-tilts-on-its-side-24/>

Note speed of orbiting at different phases of cycle, due to Kepler's laws

Milankovitch forcing: precession of orbit and of axis of rotation



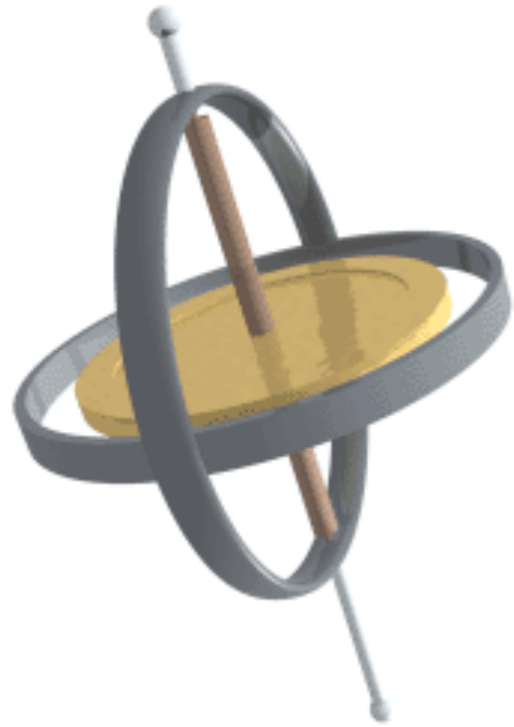
[https://commons.wikimedia.org/wiki/
File:Gyroscope_precession.gif](https://commons.wikimedia.org/wiki/File:Gyroscope_precession.gif)



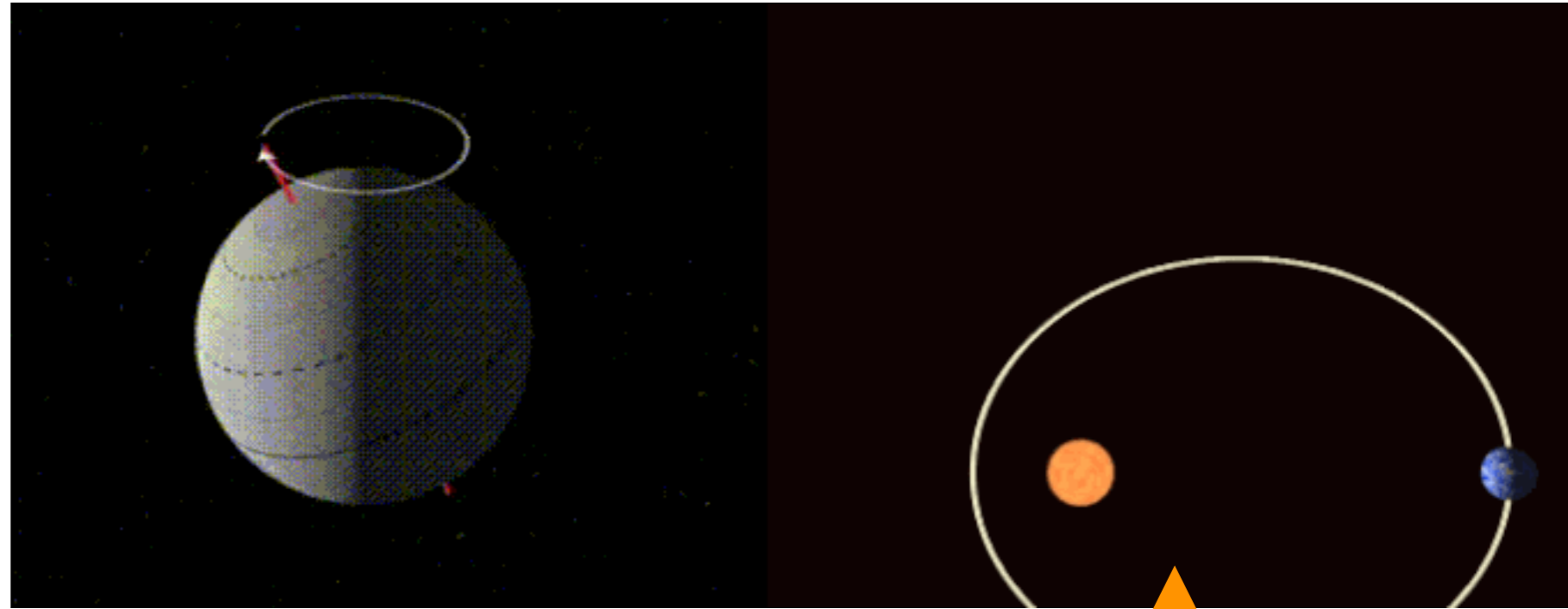
Rotational precession [left] and orbital precession [right]. [Robert Simmon / NASA; WillowW / Wikimedia Commons] downloaded from <https://www.technologyshout.com/there-is-a-new-hypothesis-about-how-uranus-tilts-on-its-side-24/>

Note speed of orbiting at different phases of cycle, due to Kepler's laws

Milankovitch forcing: precession of orbit and of axis of rotation



[https://commons.wikimedia.org/wiki/
File:Gyroscope_precession.gif](https://commons.wikimedia.org/wiki/File:Gyroscope_precession.gif)



Rotational precession [left] and orbital precession [right]. [Robert Simmon / NASA; WillowW / Wikimedia Commons]
downloaded from <https://www.technologyshout.com/there-is-a-new-hypothesis-about-how-uranus-tilts-on-its-side-24/>

Note speed of orbiting at different phases of cycle, due to Kepler's laws

Milankovitch forcing: precession

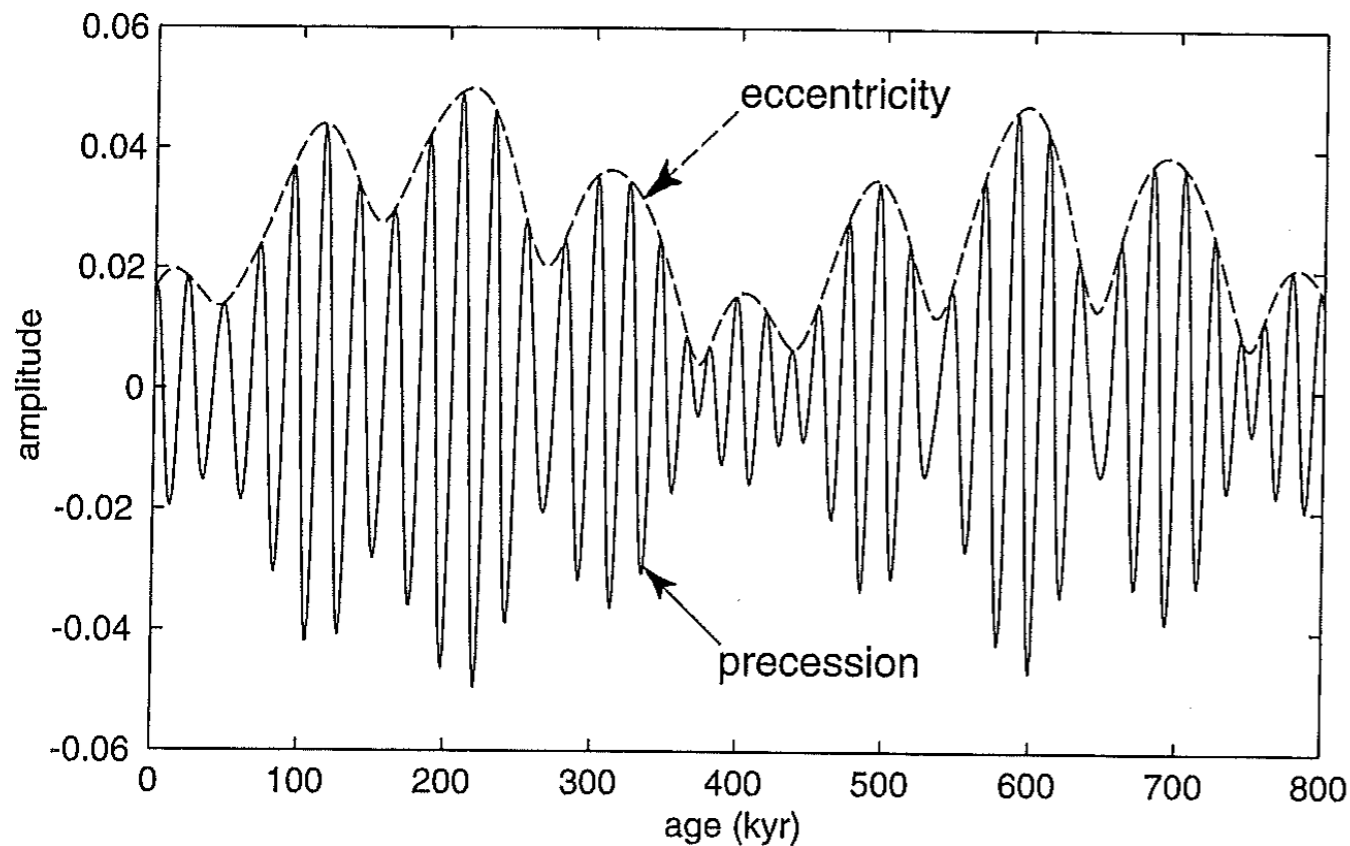


Fig. 2.10. Precession parameter p and eccentricity.

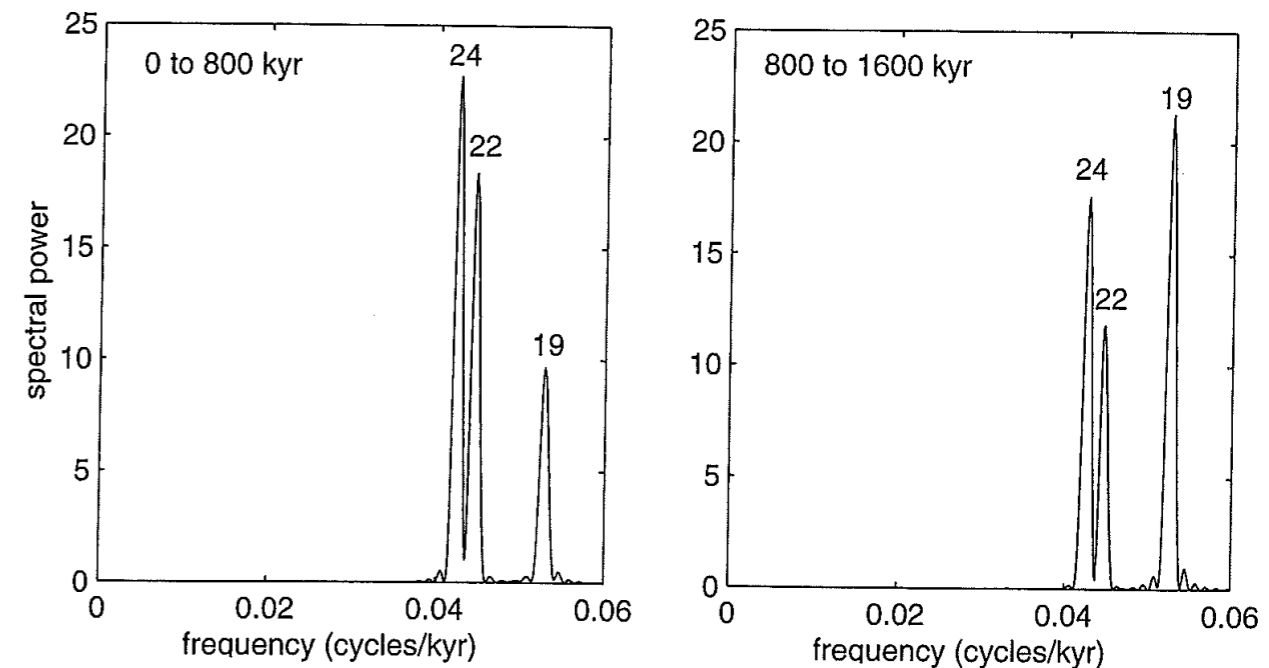


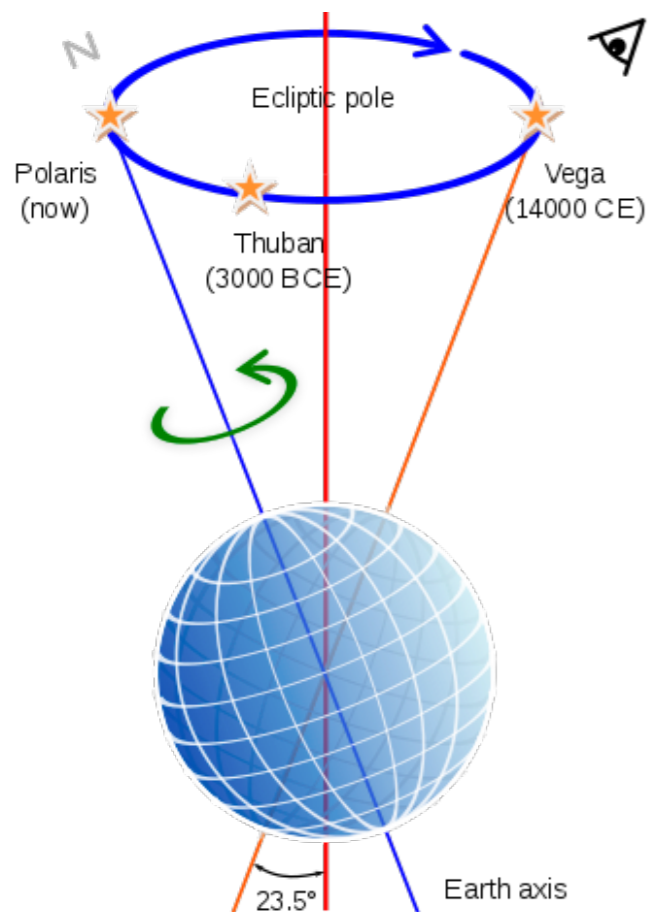
Fig. 2.11. Spectrum of precession.

$$p = e \sin \omega_M$$

climate precession parameter

precession: due to moon & sun pulling on equatorial bulge of Earth

Precession effect is anti-symmetric with respect to seasons & hemispheres; annual average of precession vanishes at each latitude; precession has no effect when eccentricity is zero (circular orbit); Paillard, 2001: “energy received at a given latitude & between two given orbital positions does not depend on the climatic precession. However, the time necessary for the Earth to move between these two positions (e.g., length of summer season) changes w/climatic precession. The insolation — amount of energy received per unit time — therefore changes w/climatic precession, through length of seasons.”



Milankovitch forcing: precession+eccentricity

In-class workshop

back of the envelope calculation to show that in spite of the small eccentricity, the combined effect of eccentricity and precession on summer insolation is very significant

- Assume a (moderately large) Earth orbit eccentricity of 0.03
- What are the configurations of the precession angle that lead to a maximum summer insolation at 65N and to a minimum summer insolation there?
- Using 1361 W/m^2 as the mean top-of-atmosphere solar constant and the average Earth–Sun distance as 1 astronomical unit (AU) $\approx 149,597,870.7 (150 \cdot 10^6) \text{ km}$, calculate the maximum top of the atmosphere summer insolation at each of the two configurations.
- What is the difference in daily-averaged insolation at 65N.
- How big is the difference in W/m^2 between the two configurations? As a percent fraction of the weaker insolation?

[solution: presenter notes]

Milankovitch forcing: obliquity

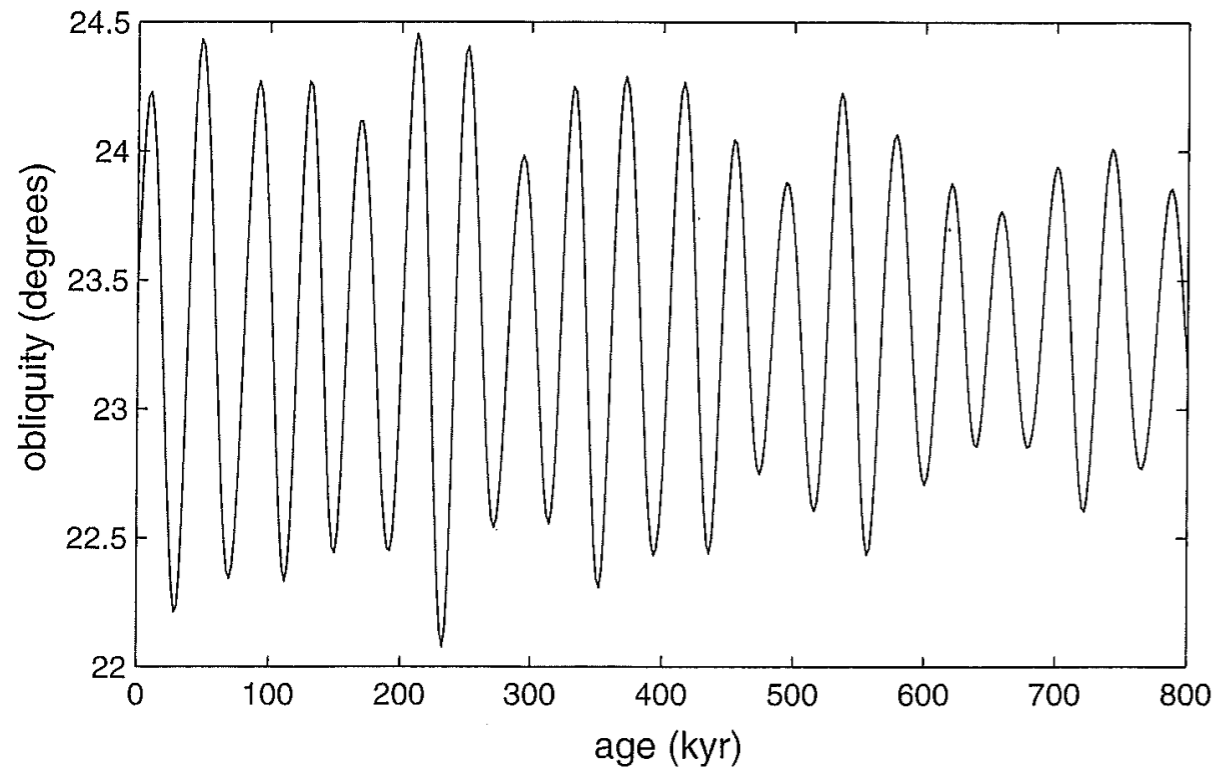


Fig. 2.12. Obliquity.

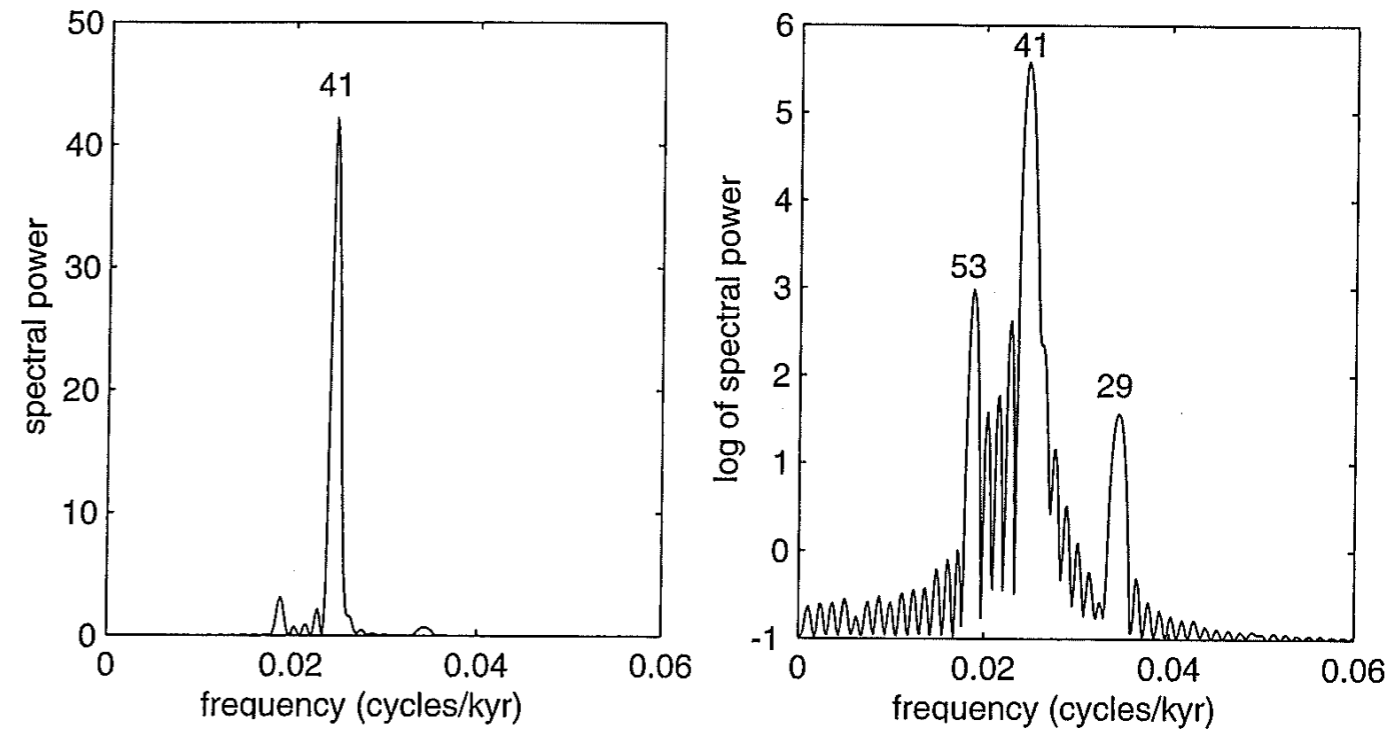
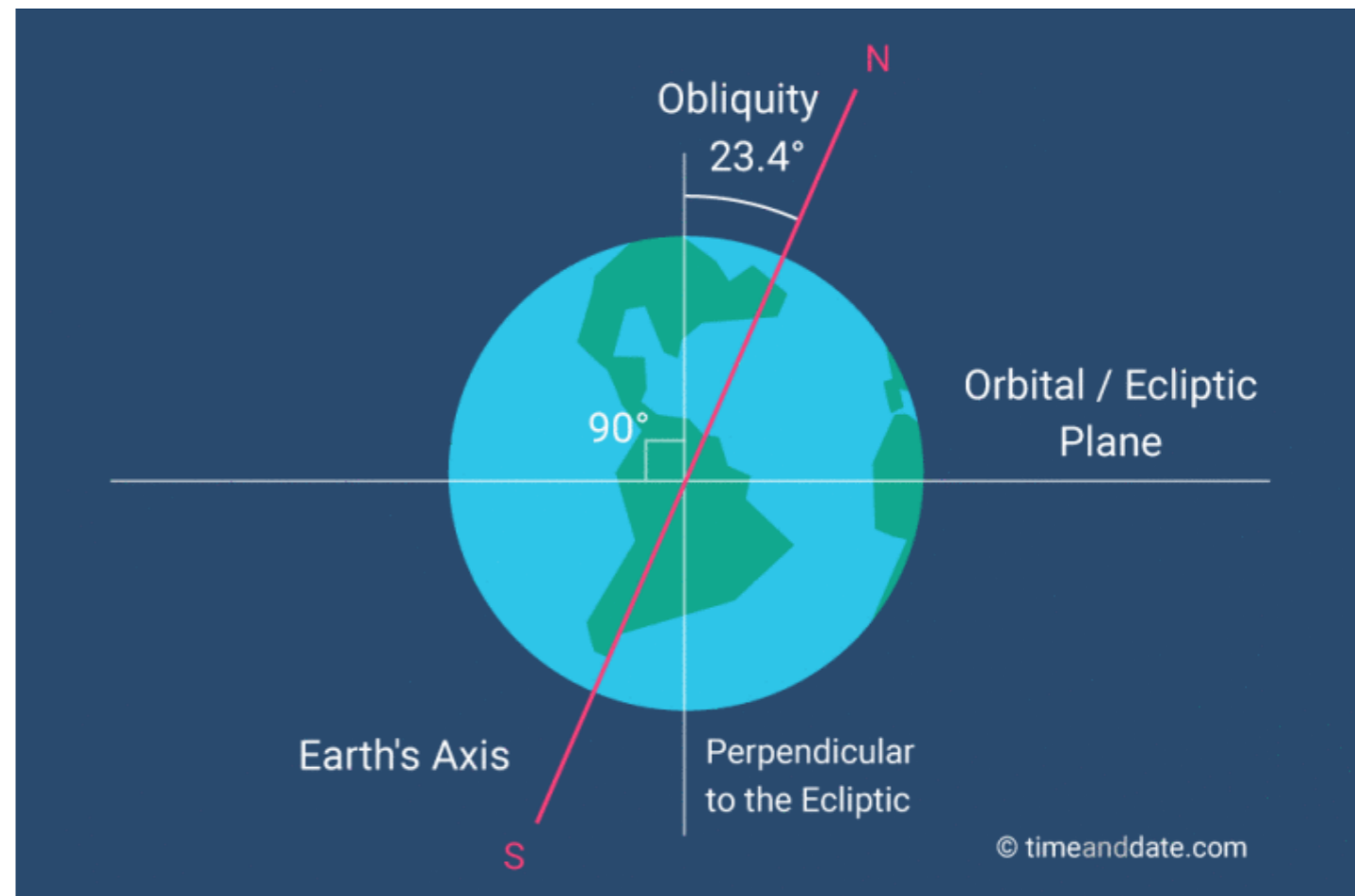


Fig. 2.13. Spectrum of obliquity, linear and log plots.

Muller & MacDonald 2002

Obliquity does affect the annual mean insolation at a given latitude, but not the global average. ➔ Antisymmetric effect on the N/S hemispheres.

Mechanism: Larger obliquity leads to more radiation at the poles in summers, but still none at winter, so more generally high latitude annual insolation depends on obliquity.



Milankovitch forcing: combining precession, obliquity, eccentricity

July insolation power
at different latitudes

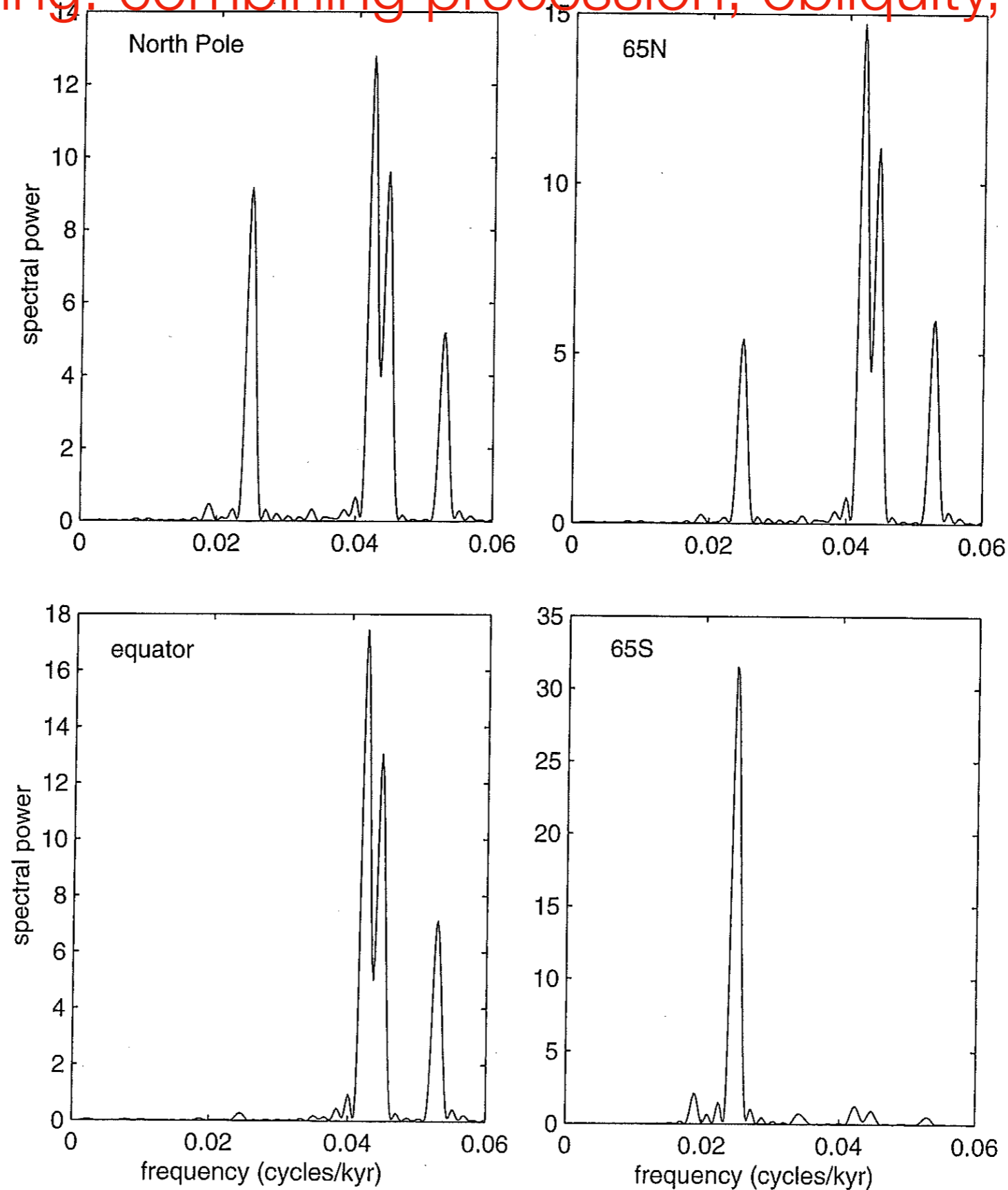
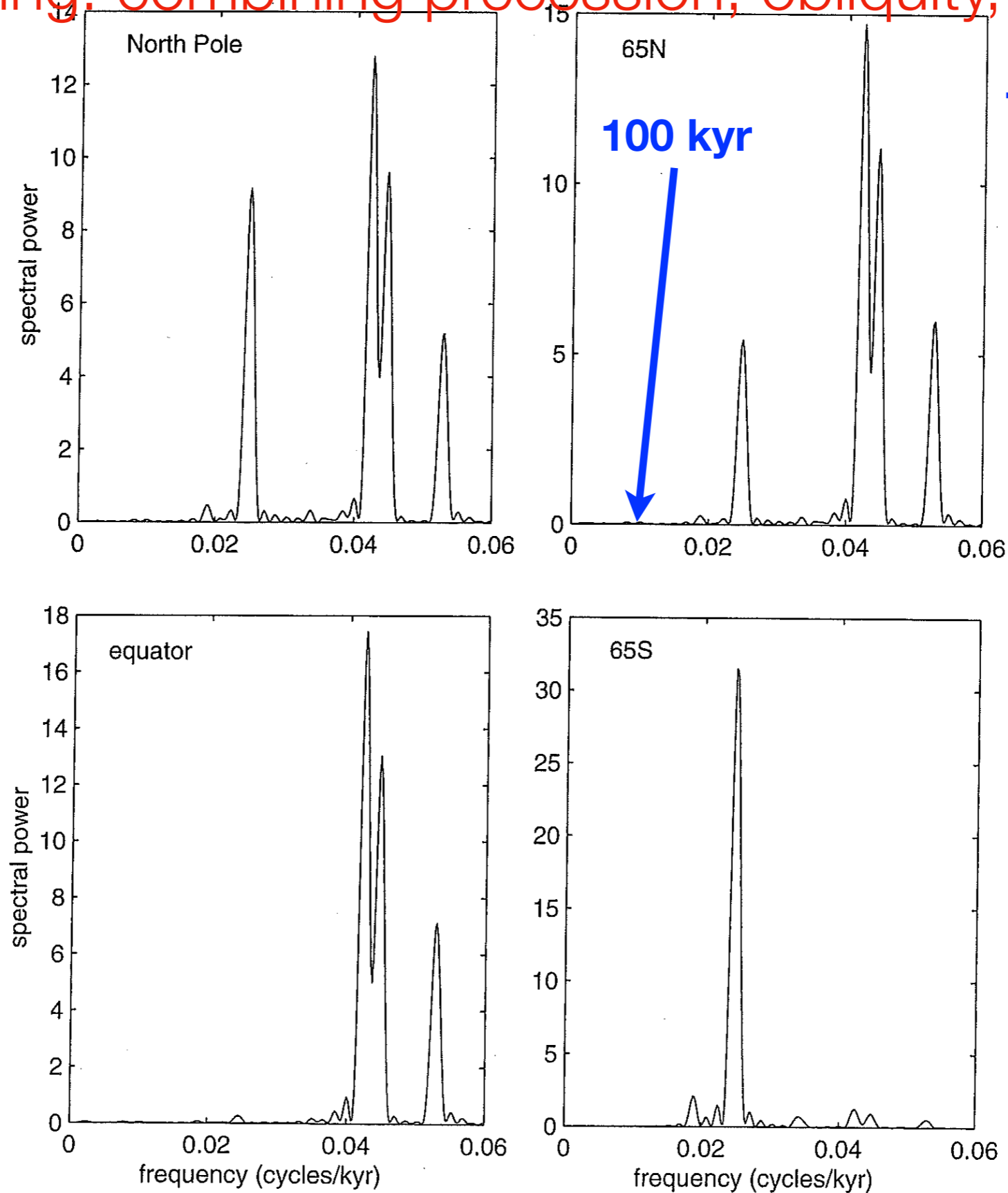


Fig. 2.14. Spectra of July insolation, at latitudes 90N, 65N, 0 and 65S.

Milankovitch forcing: combining precession, obliquity, eccentricity

July insolation power at different latitudes



The Milankovitch Paradox!

Fig. 2.14. Spectra of July insolation, at latitudes 90N, 65N, 0 and 65S.

notes: parabolic ice sheet

(simple version only, 1-WH-lect_08_2001.pdf p 100

use next slide)

Parabolic ice sheet profile

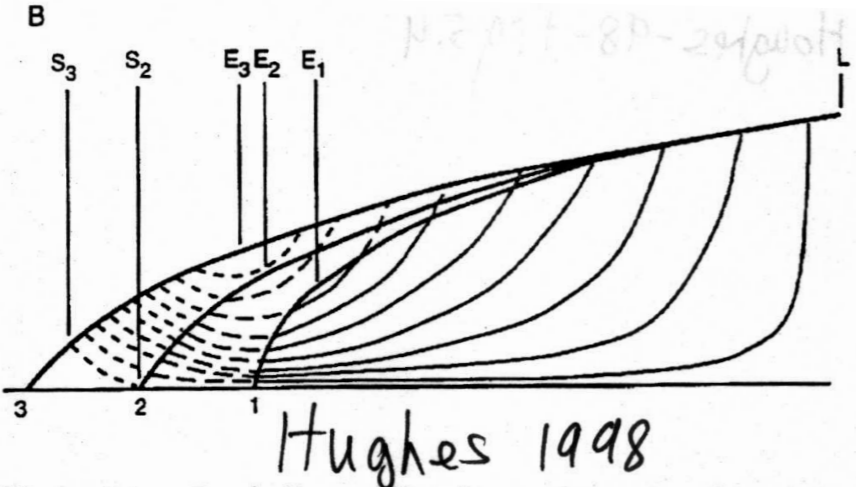
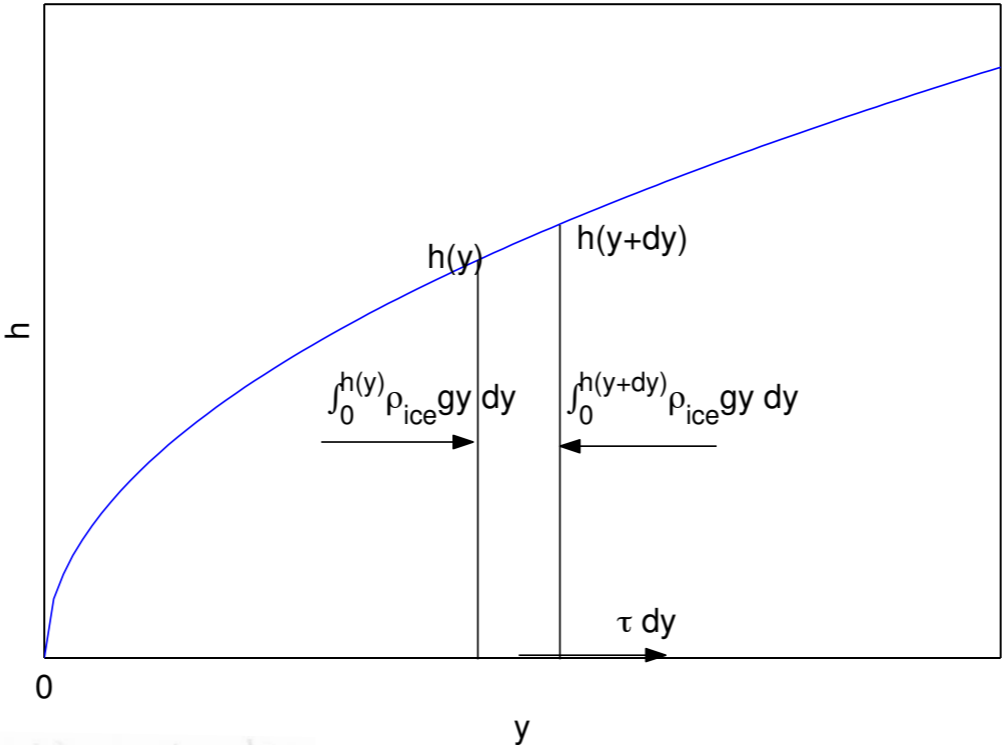


Figure 5.4: Parabolic profiles for a plastic ice sheet on a horizontal bed. (A) Profile 1: $\tau_0 = 105$ kPa for an advancing ice sheet. Profile 2: $\tau_0 = 66$ kPa for an equilibrium ice sheet. Profile 3: $\tau_0 = 42$ kPa for a retreating ice sheet. (B) Ice trajectories for ice sheets that are advancing (solid curves), in equilibrium (solid and broken curves), and retreating (solid, broken, and dashed curves). Equilibrium points are E_1 for advancing, E_2 for equilibrium, and E_3 for retreating ice sheets. Stagnation points are S_2 for equilibrium and S_3 for retreating ice sheets.

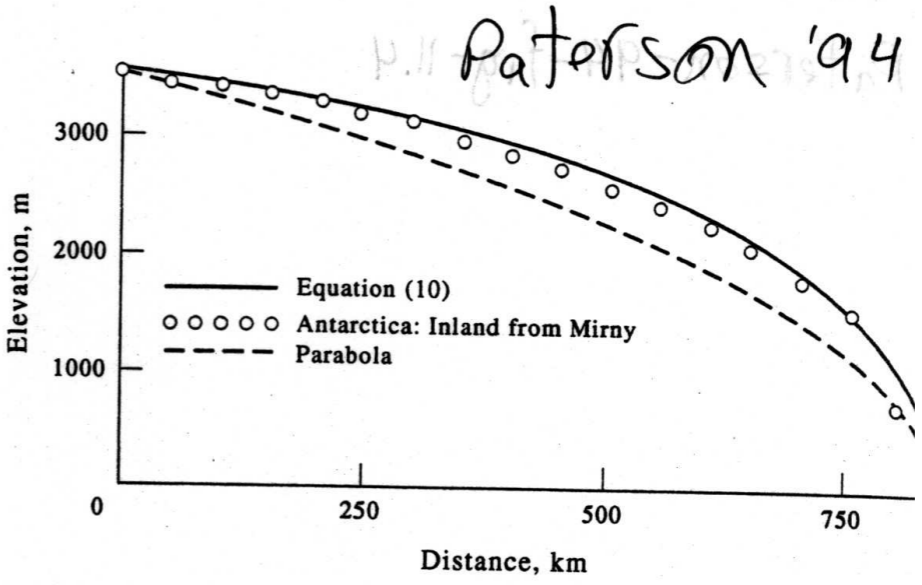
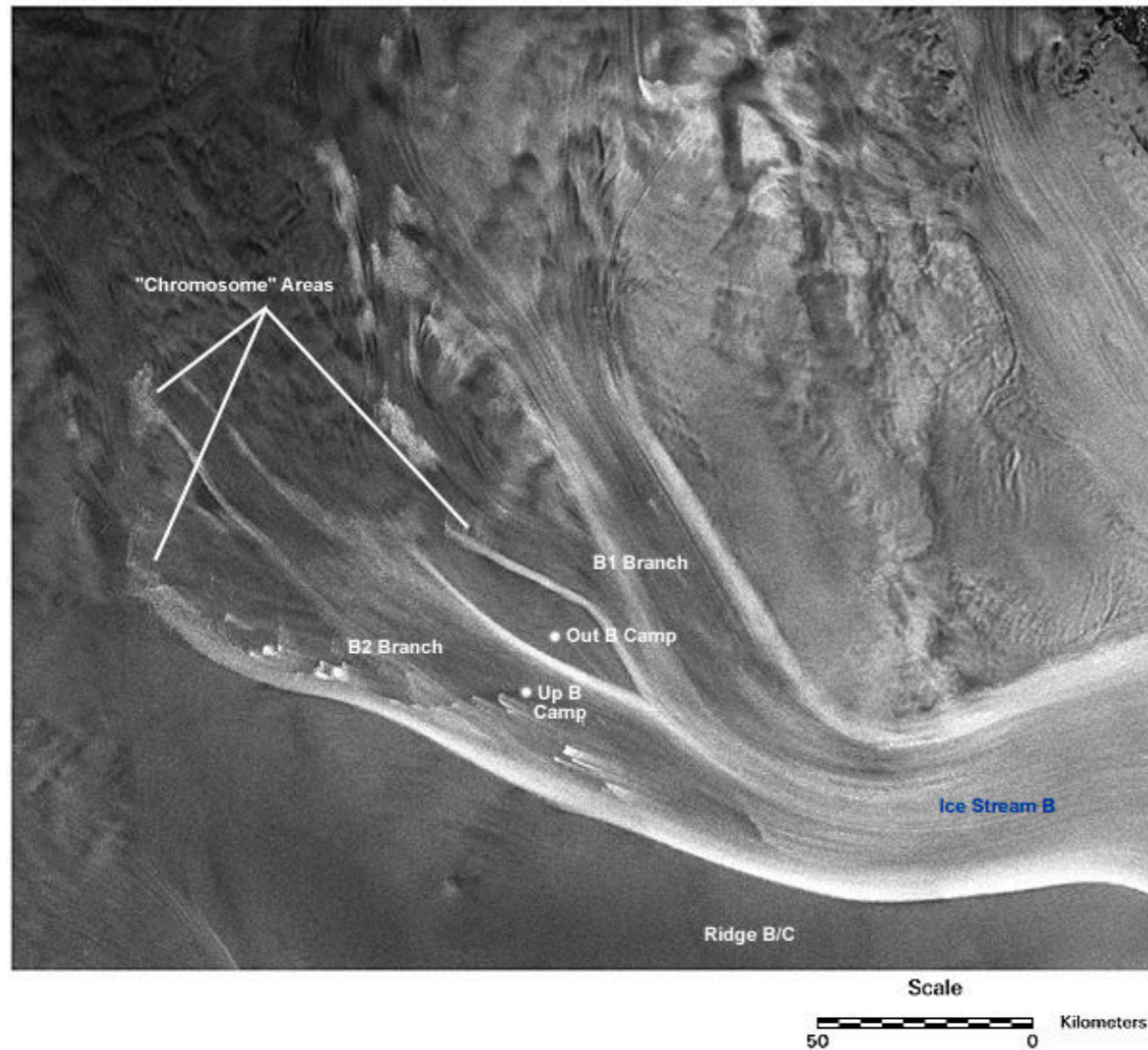


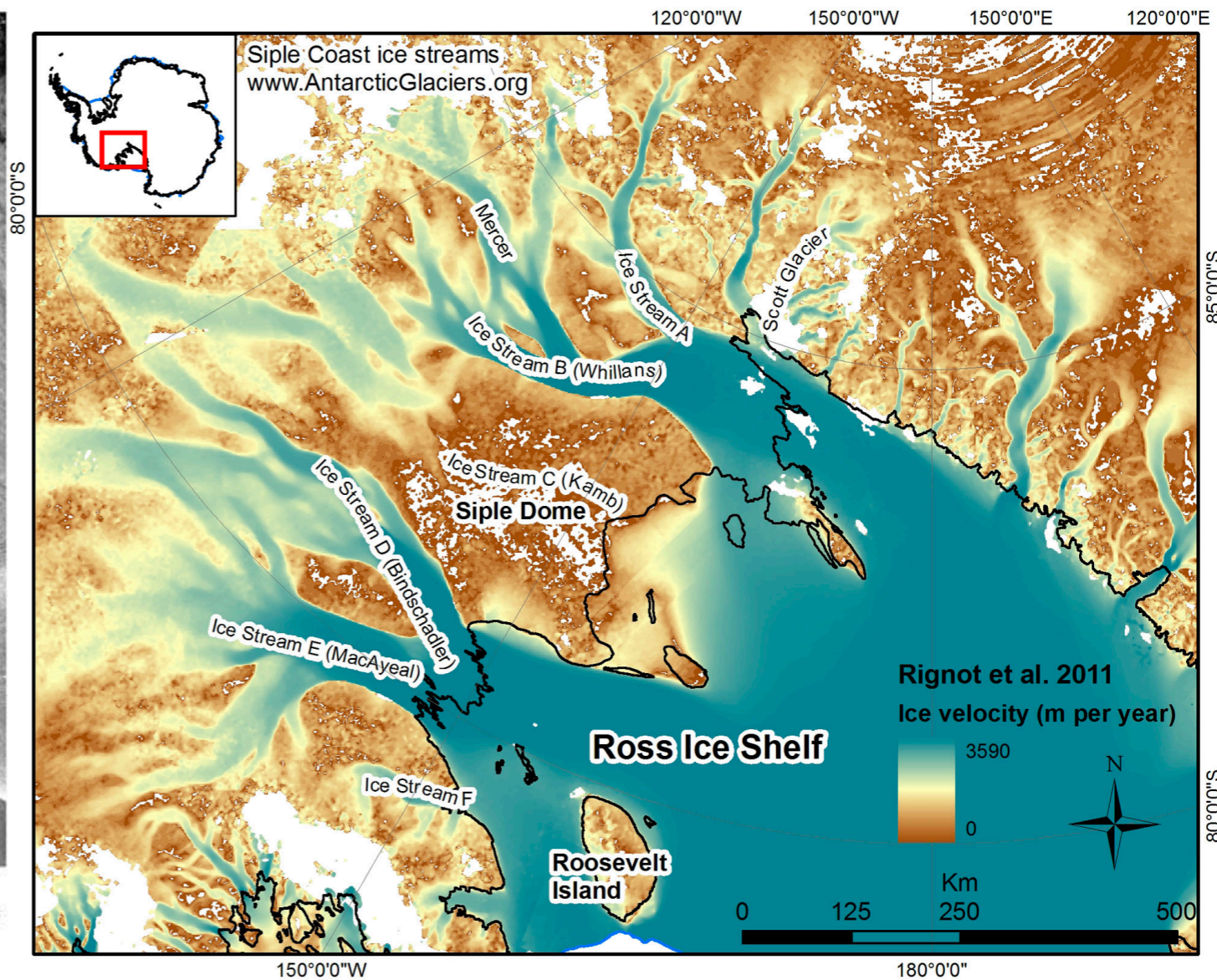
FIG. 11.4. Profile of Antarctic Ice Sheet inland from Mirny compared with theoretical profiles. Data from Vialov (1958).

Ice streams, calving, ablation



ice stream B, Antarctica

<https://nsidc.org/support/faq/what-features-can-i-detect-ramp-amm-1-sar-image-mosaic-antarctica>



Rignot et al 2011

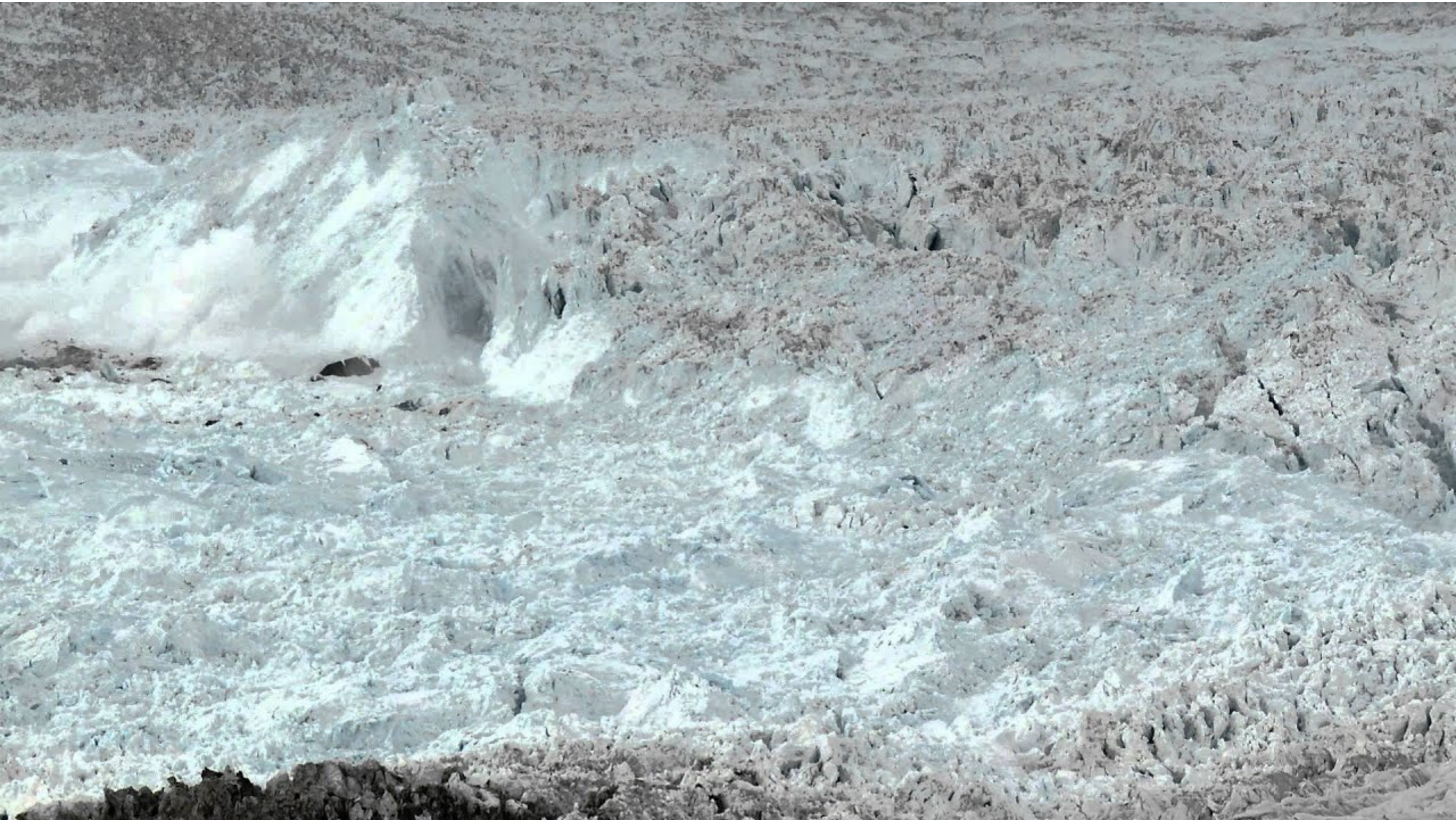
Ablation, ice flow Antarctic ice streams



Ablation, ice flow Antarctic ice streams

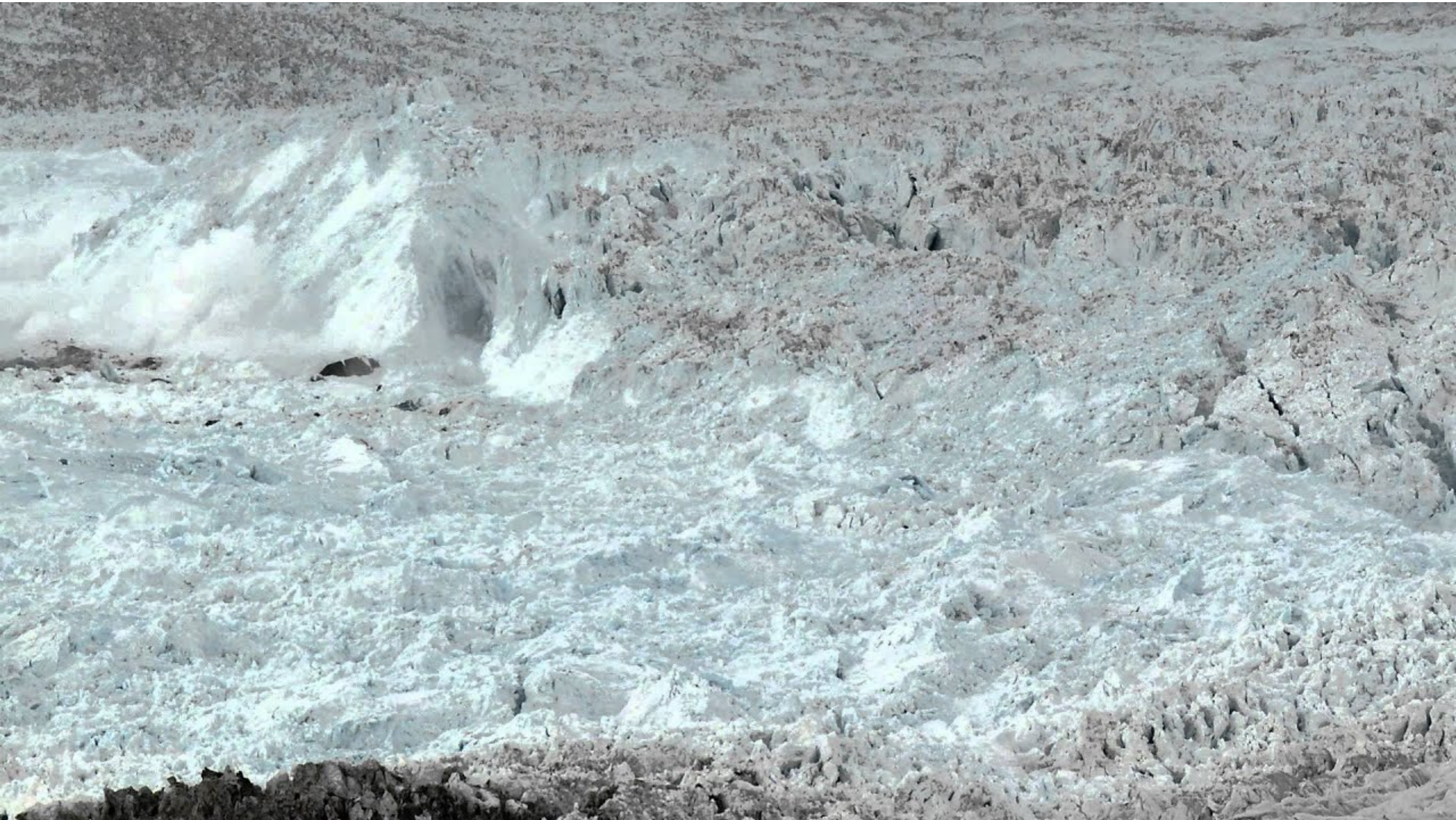


Greenland Calving event, “Chasing ice” film



<https://www.youtube.com/watch?v=hC3VTgIPoGU>

Greenland Calving event, “Chasing ice” film

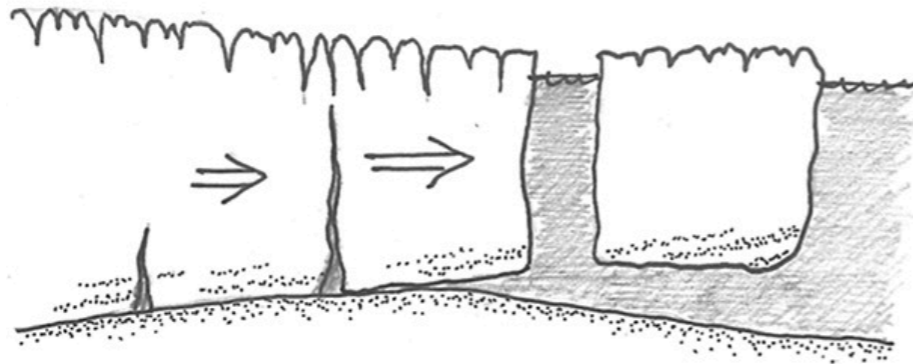


<https://www.youtube.com/watch?v=hC3VTgIPoGU>

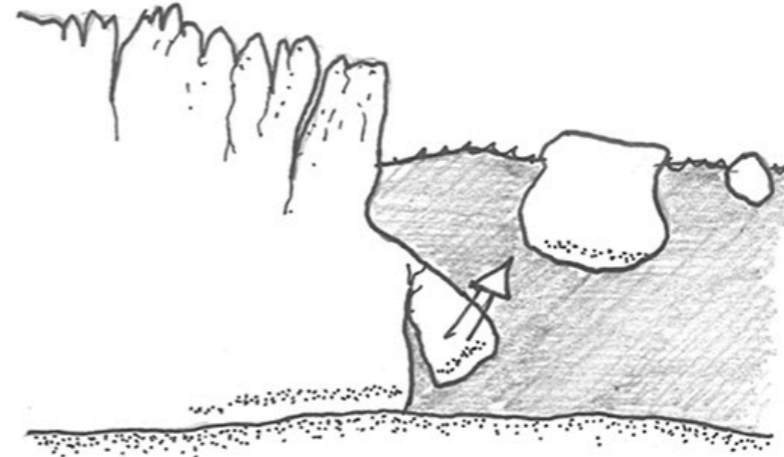
Ablation

Calving: role of buoyancy forces

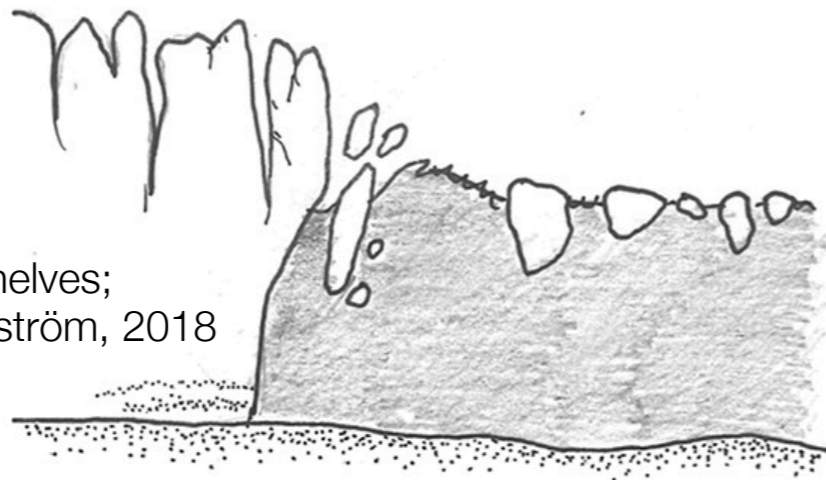
A: Longitudinal extension



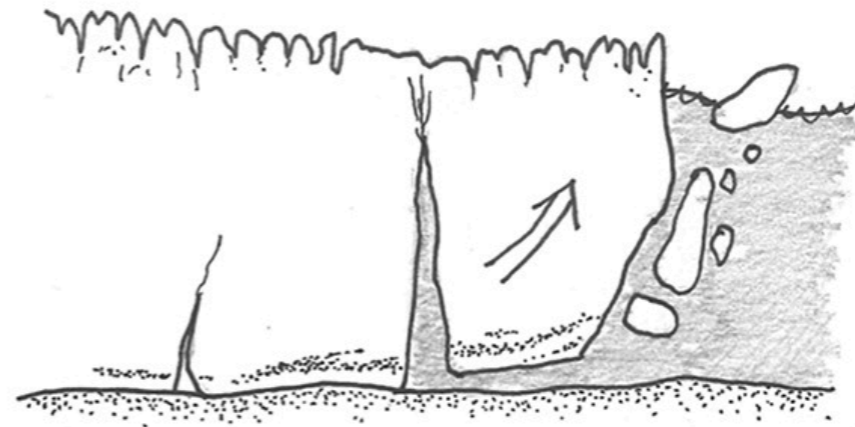
C: Buoyant calving - ice foot



B: Melt-undercutting



D: Buoyant calving - full thickness



Calving glaciers and ice shelves;
Douglas I. Benn, Jan A. Åström, 2018

Figure 2. A selection of key calving styles: **(a)** rifting due to longitudinal extension, **(b)** collapse of overhang following undercutting by subaqueous melt, **(c)** buoyant calving: release of a protruding 'ice foot' below the waterline and **(d)** buoyant calving: uplift of a super-buoyant glacier tongue.

Temperature precipitation feedback

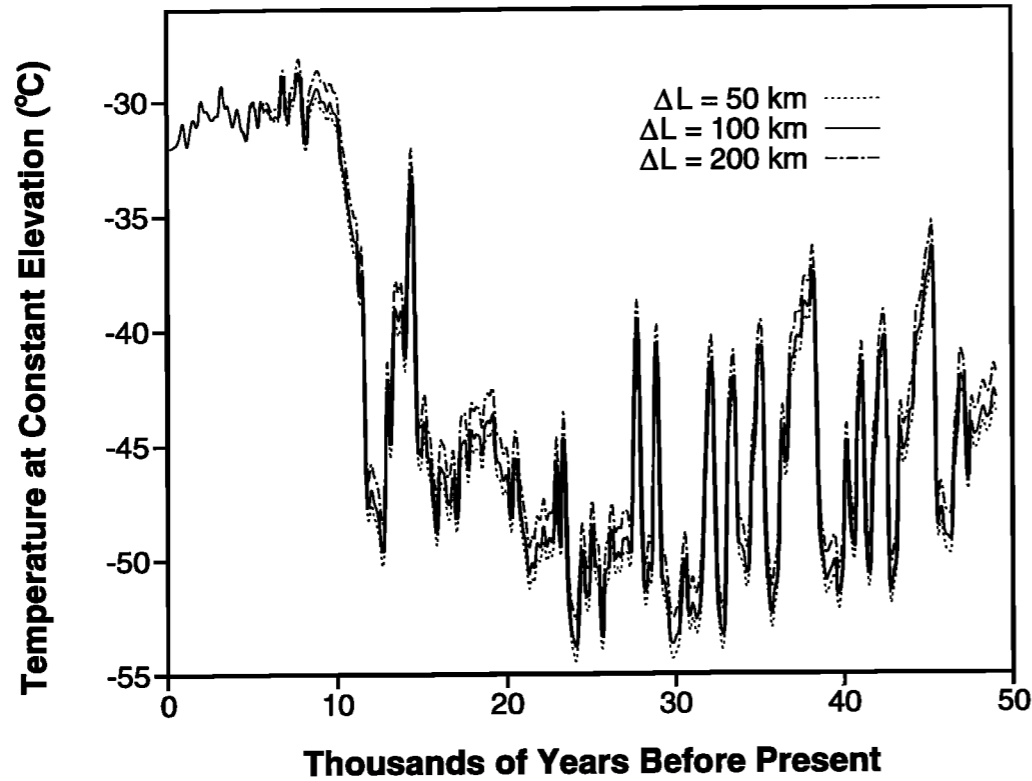


Figure 3. Temperature history according to calibrated isotope curve, corrected for elevation changes. The data have been smoothed with a 250-year triangular filter so that the effect of different elevation corrections, corresponding to different marginal retreat distances, can be seen.

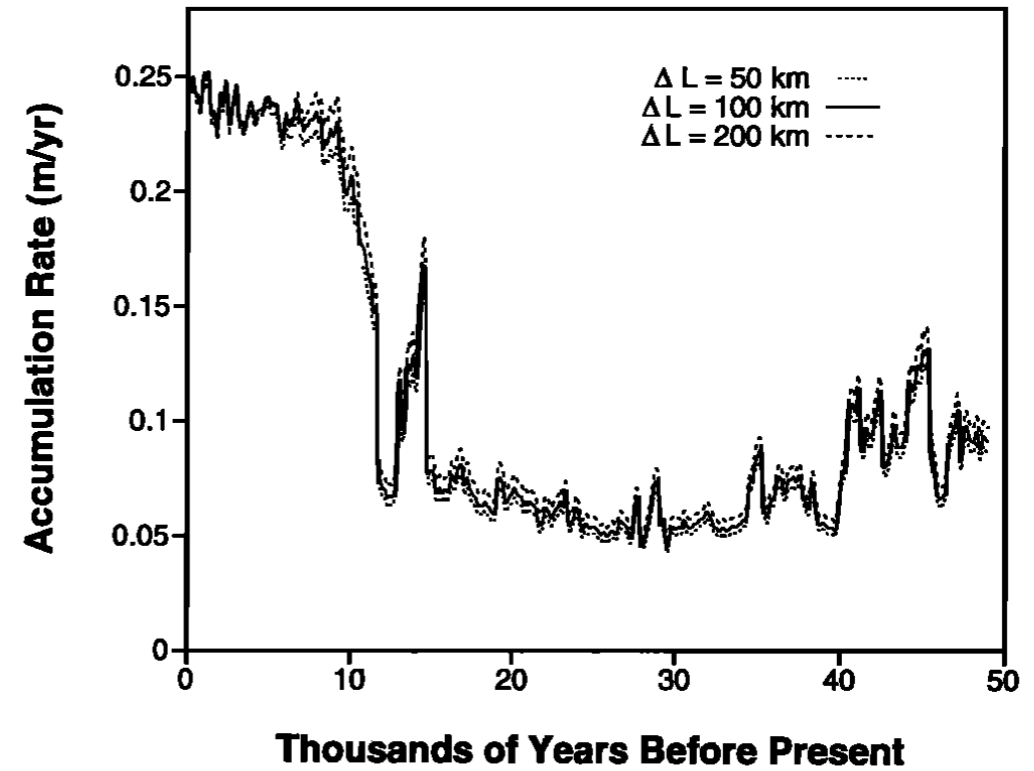


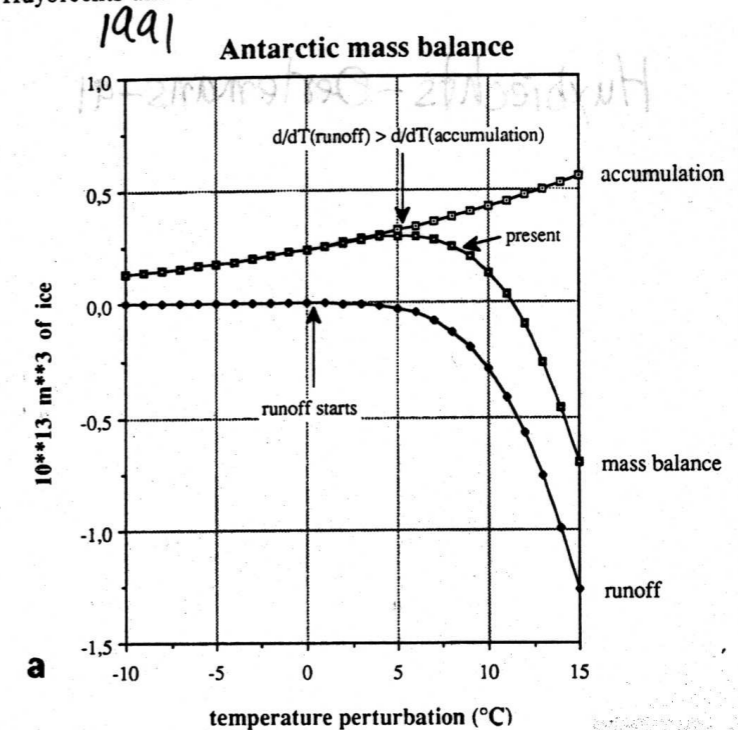
Figure 5. Accumulation rate histories for different marginal retreat distances.

Will greenhouse warming lead to Northern Hemisphere ice-sheet growth?

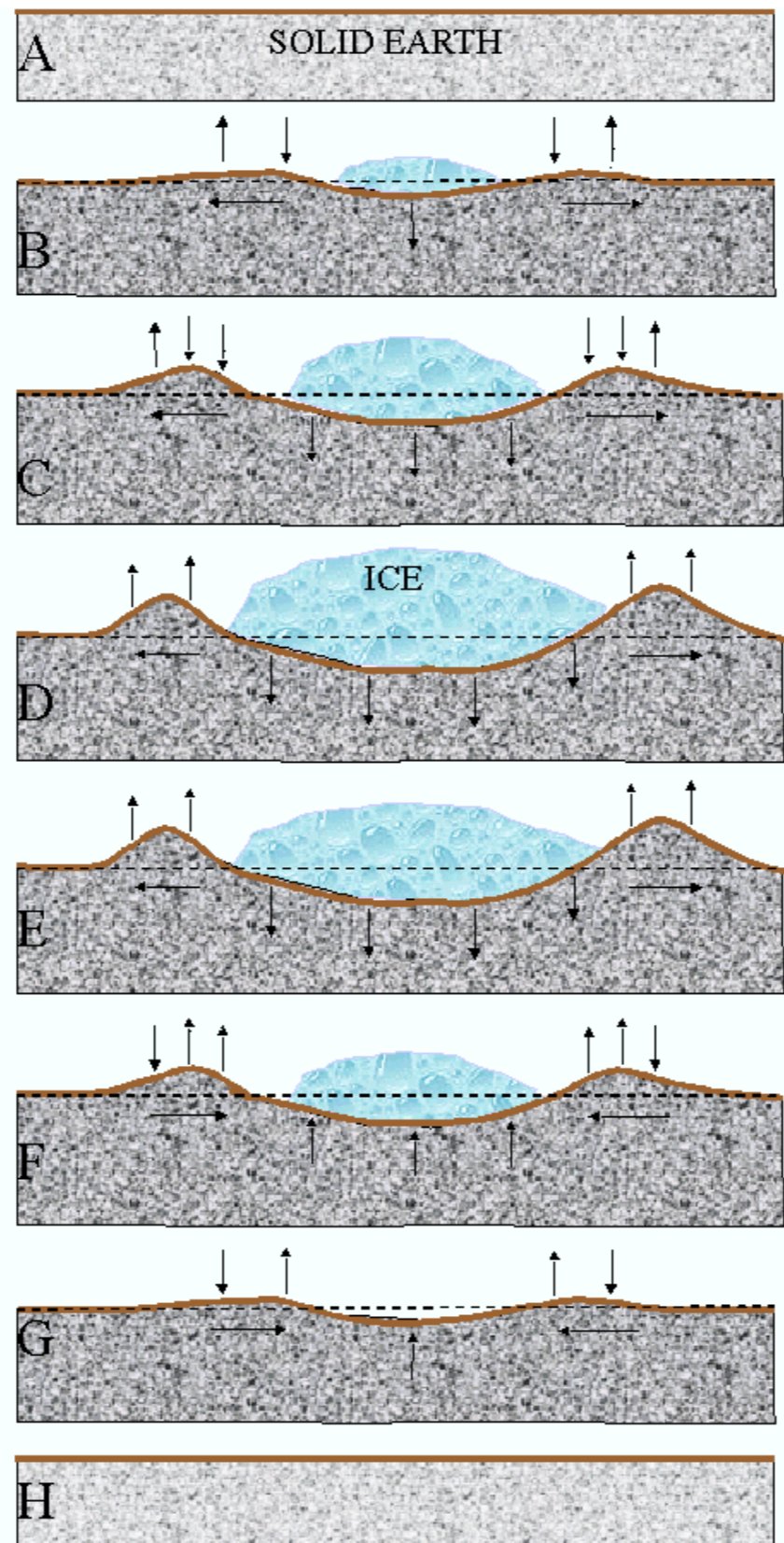
Gifford H. Miller* & Anne de Vernal† 1992

* Center for Geochronological Research, INSTAAR, University of Colorado, Boulder, Colorado 80309-0450, USA
 † GEOTOP, Université du Québec à Montréal, CP 8888, Succursale "A", Montréal, Québec, Canada, H3C 3P8

Huybrechts and Oerlemans:



isostatic adjustment



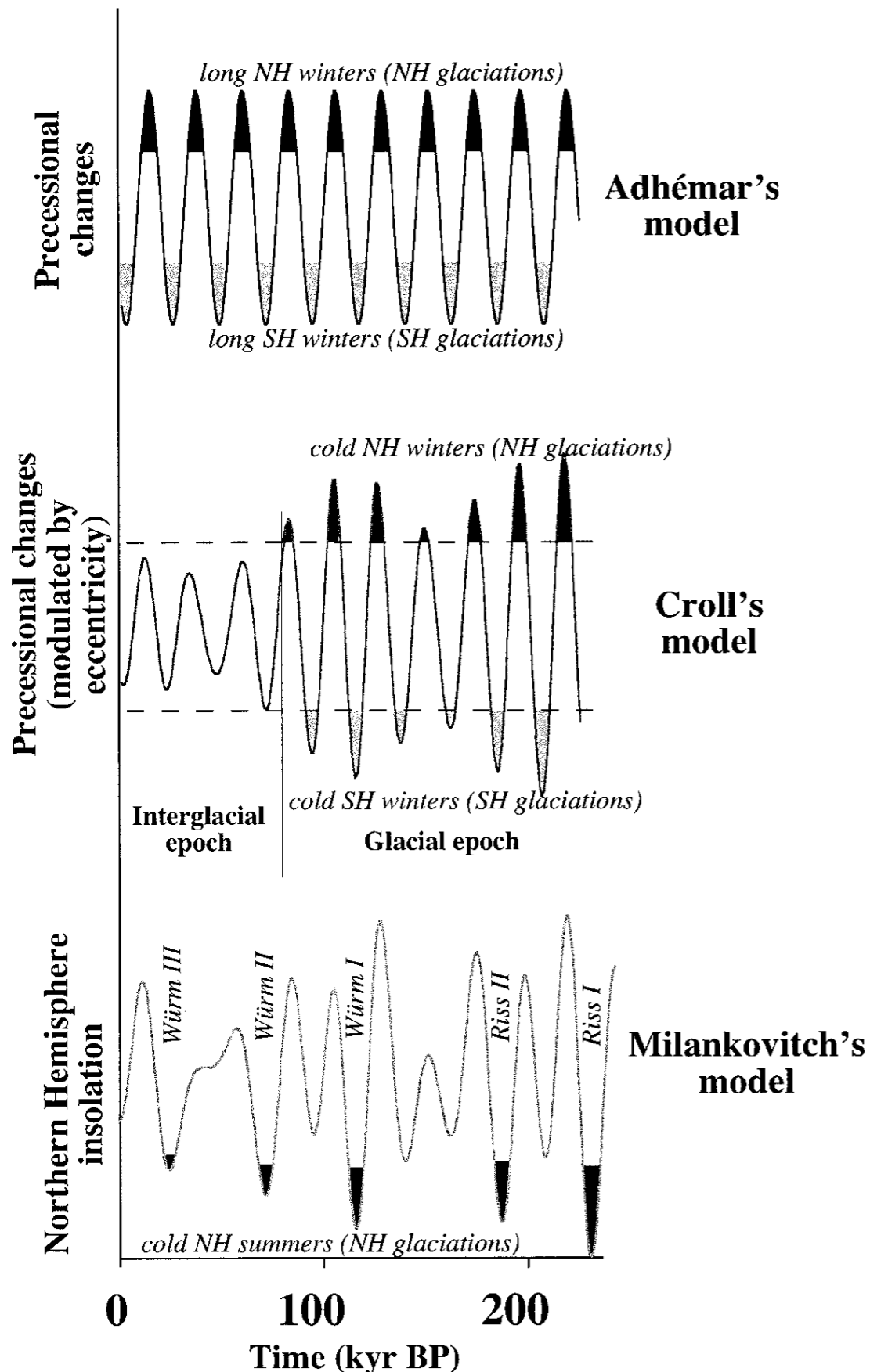


Figure 1. The ice ages according to **Adhemar [1842]**, **Croll [1875]**, and **Milankovitch [1941]**. Adhemar was aware only of the precession of equinoxes, and he related the glacial ages to the lengths of the seasons. Croll benefited from the advances in astronomy and was aware of changes in the other astronomical parameters, though he could not compute the obliquity changes. In his view, the interglacial epoch is associated with small eccentricity and therefore with small precessional changes. **Milankovitch** was the first to integrate the effect of all astronomical parameters and to compute explicitly the insolation at the top of the atmosphere. He **understood that summer, not winter, was the critical season**. His insolation minima were associated with the major alpine glacier advances recorded by geological evidence.

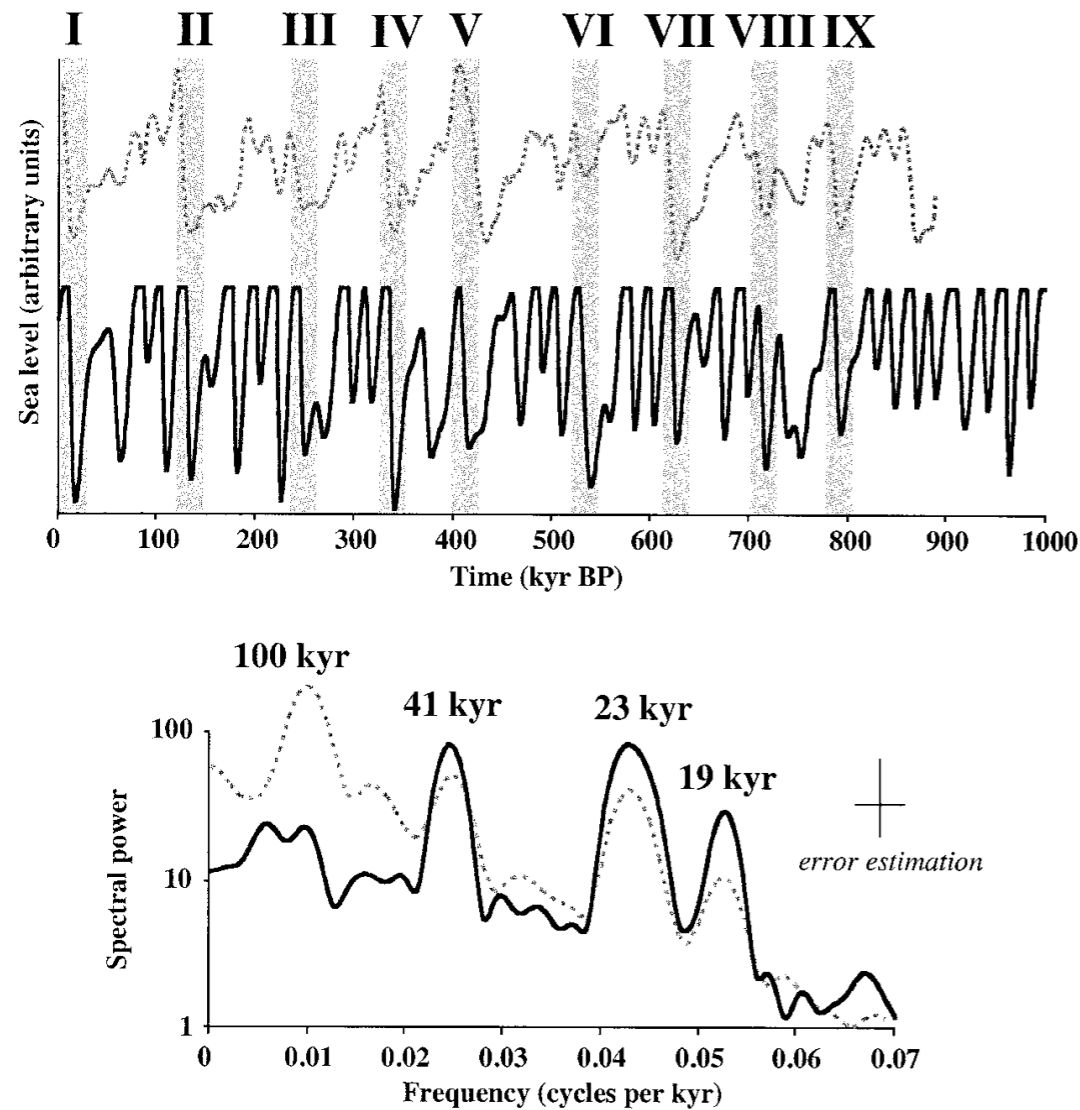


Figure 9. Results from the **Calder [1974] model**. The threshold i_0 is equal to 502 W m^2 , and the ratio k_A/k_M is chosen equal to 0.22. The forcing i is the summer solstice insolation at 65°N [Laskar, 1990]. The result is very sensitive to these choices. The agreement with the record is quite poor, but this crude model still predicts the major transitions at the right time, a feature that many, more sophisticated models do not reproduce well. An isotopic record is given here for comparison [Bassinot et al., 1994b].

$$\frac{dV}{dt} = -k(i - i_0),$$

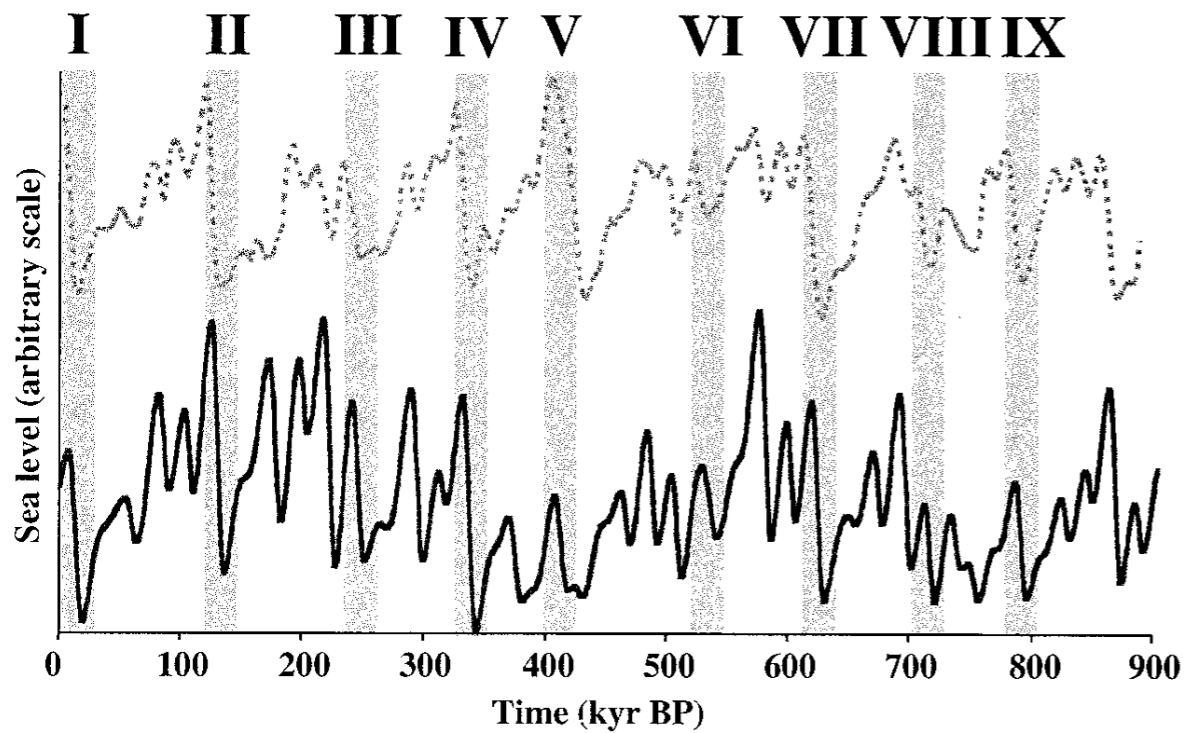
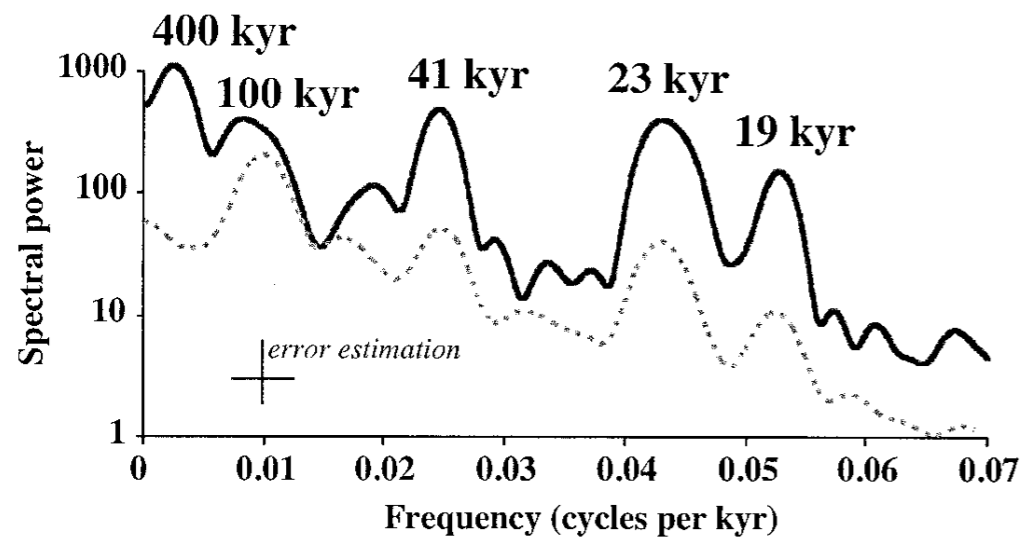
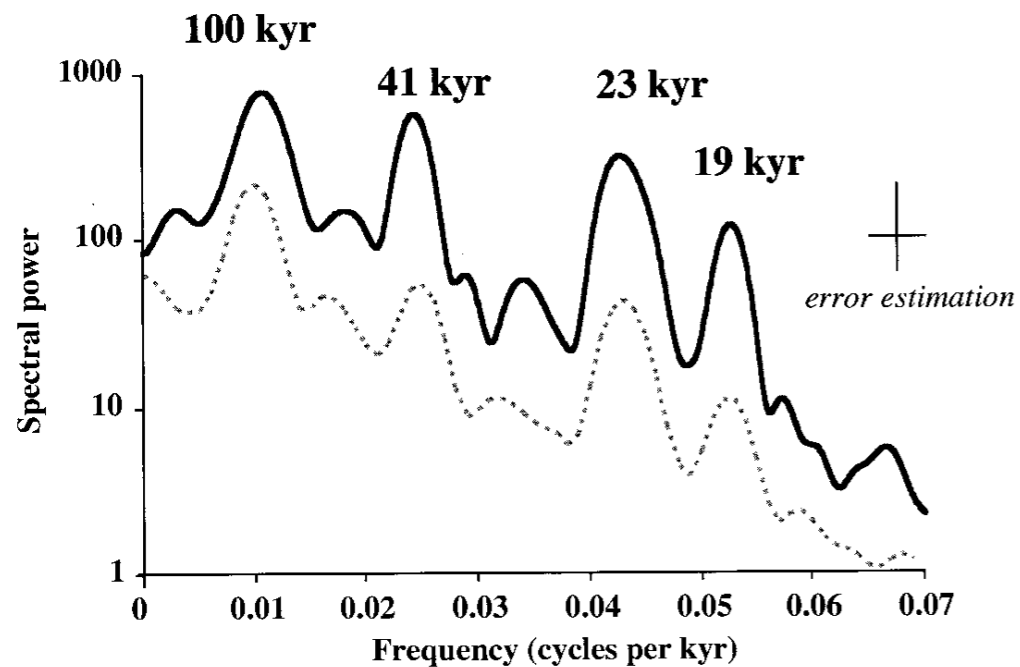
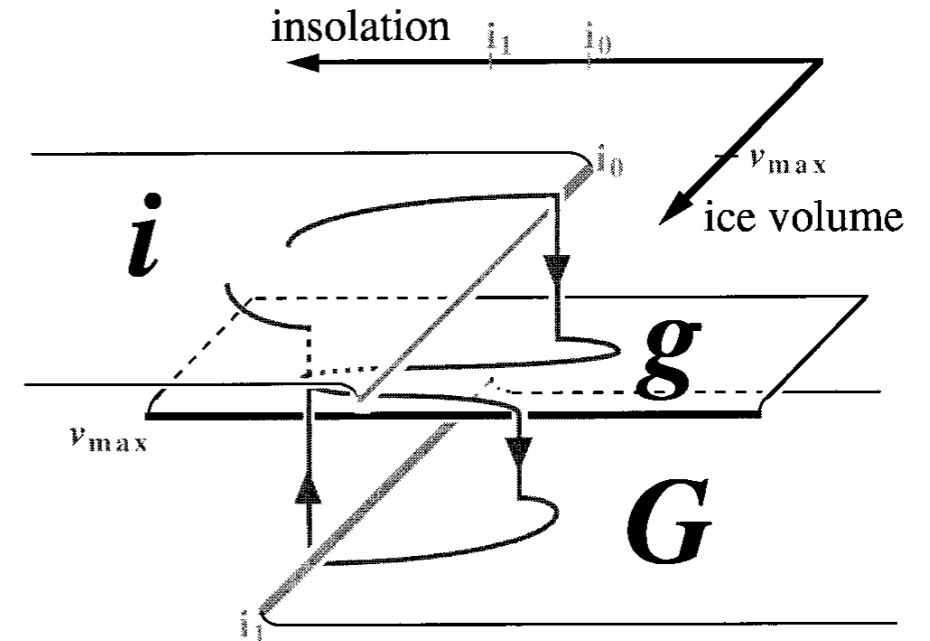
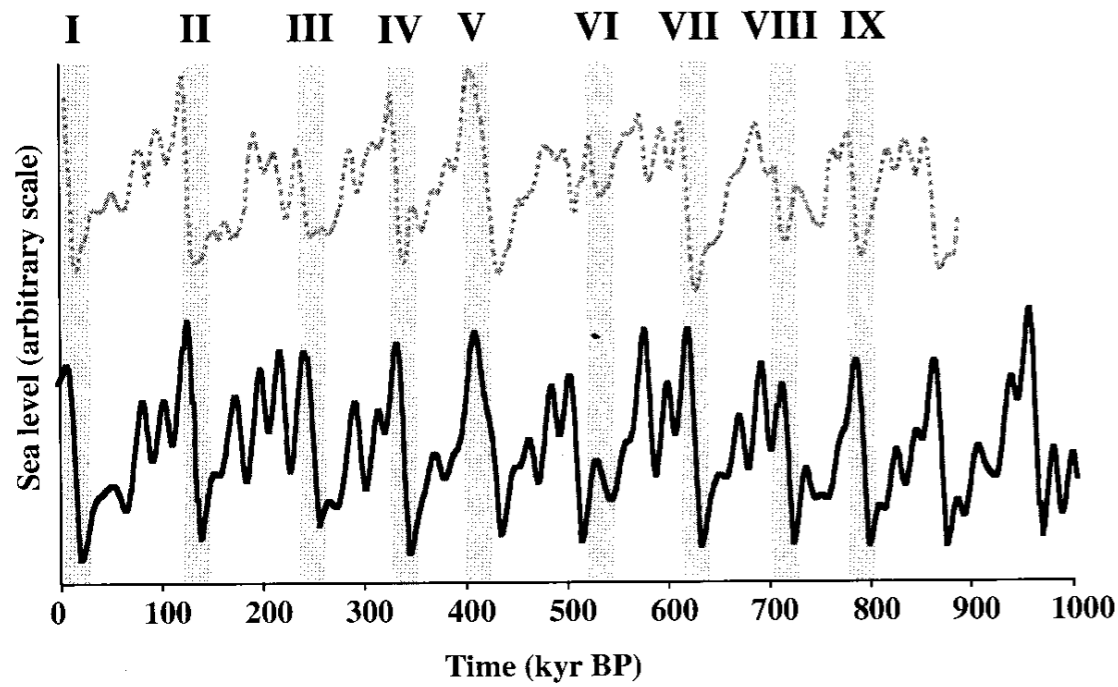


Figure 10. Same as Figure 9, but for the Imbrie and Imbrie [1980] model. The forcing i is the summer solstice insolation at 65N. The time constants are $\tau_M=42$ kyr and $\tau_A=10$ kyr.



$$\frac{dV}{dt} = \frac{(i - V)}{\tau},$$



$$\frac{dV}{dt} = - \frac{(V_R - V)}{\tau_R} - \frac{F}{\tau_F},$$

Figure 13. Same as Figure 9, but for Paillard's [1998] model. Threshold values are $i_0=0.75$ and $i_1=0$. Time constants are $\tau_i=10$ kyr, $\tau_G=\tau_g=50$ kyr, and $\tau_F=25$ kyr. A total of about 14 model parameters.

100 kyr cycle as a nonlinear response to Milankovitch

Le-Treut and Ghil (1983): the 100 kyr cycle as a difference tone of insolation's 19k and 23k frequencies, due to nonlinear glacial dynamics.

$$\frac{1}{109} \text{kyr}^{-1} = f_1 - f_2 = \frac{1}{19} \text{kyr}^{-1} - \frac{1}{23} \text{kyr}^{-1}$$

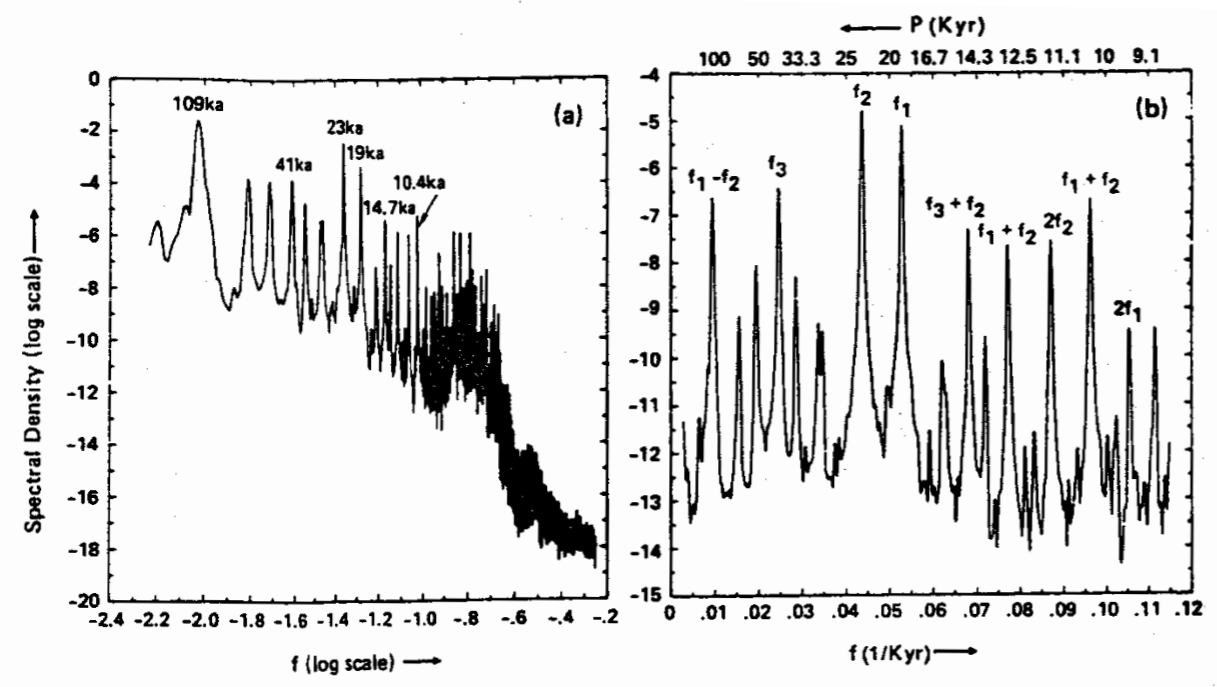


Fig. 10. Power spectrum of (a) simulated marine $\delta^{18}O$ and (b) ice-core $\delta^{18}O$ records. The periods associated with the labeled peaks in panel (b) are, from left to right, 109 kyr, 41 kyr, 23 kyr, 19 kyr, 14.7 kyr, 13 kyr, 11.5 kyr, 10.4 kyr and 9.5 kyr (from [90]).

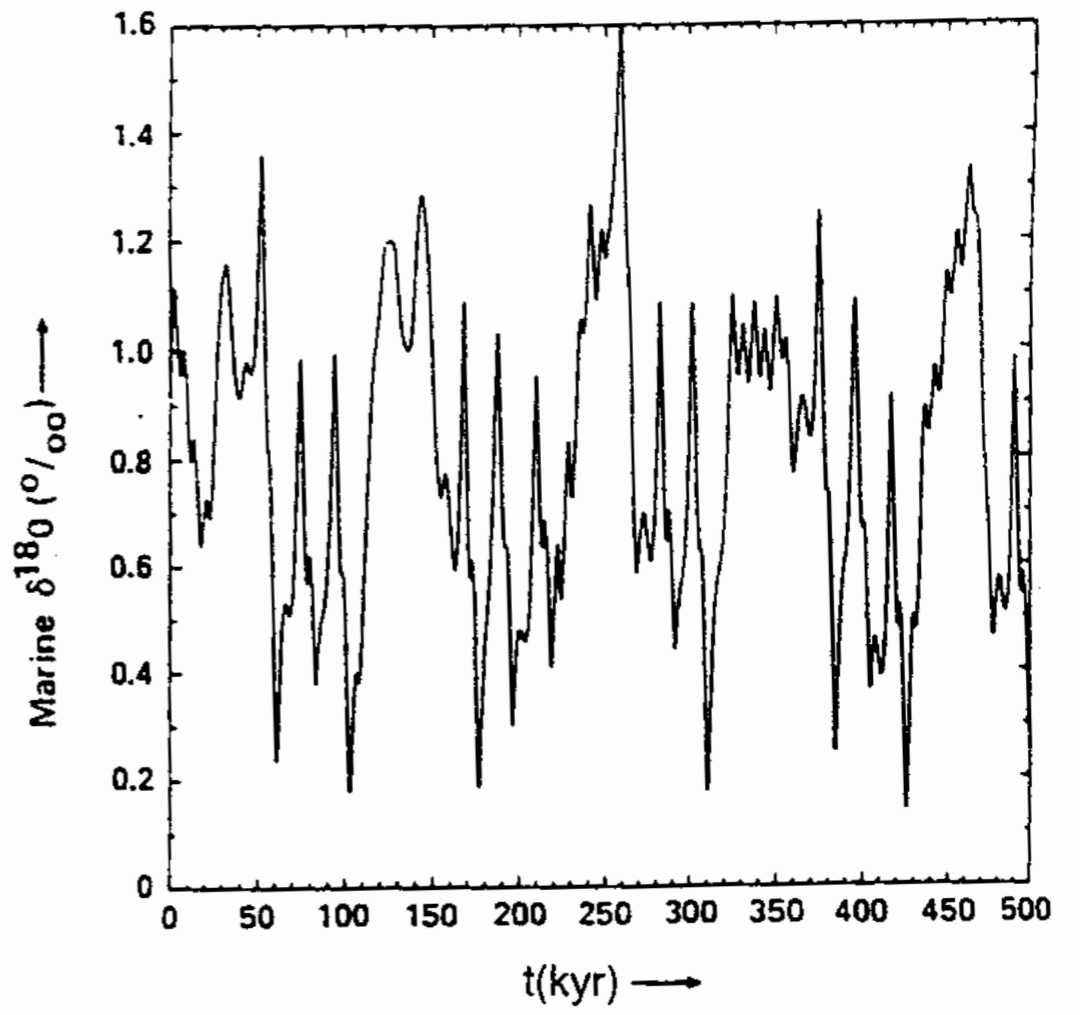


Fig. 11. Model-simulated marine $\delta^{18}O$ record; its power spectrum appears in Fig. 10a (from [90]).

Then Rial (1999):

$$\frac{1}{107} \text{kyr}^{-1} = \frac{1}{95} \text{kyr}^{-1} - \frac{1}{826} \text{kyr}^{-1}$$

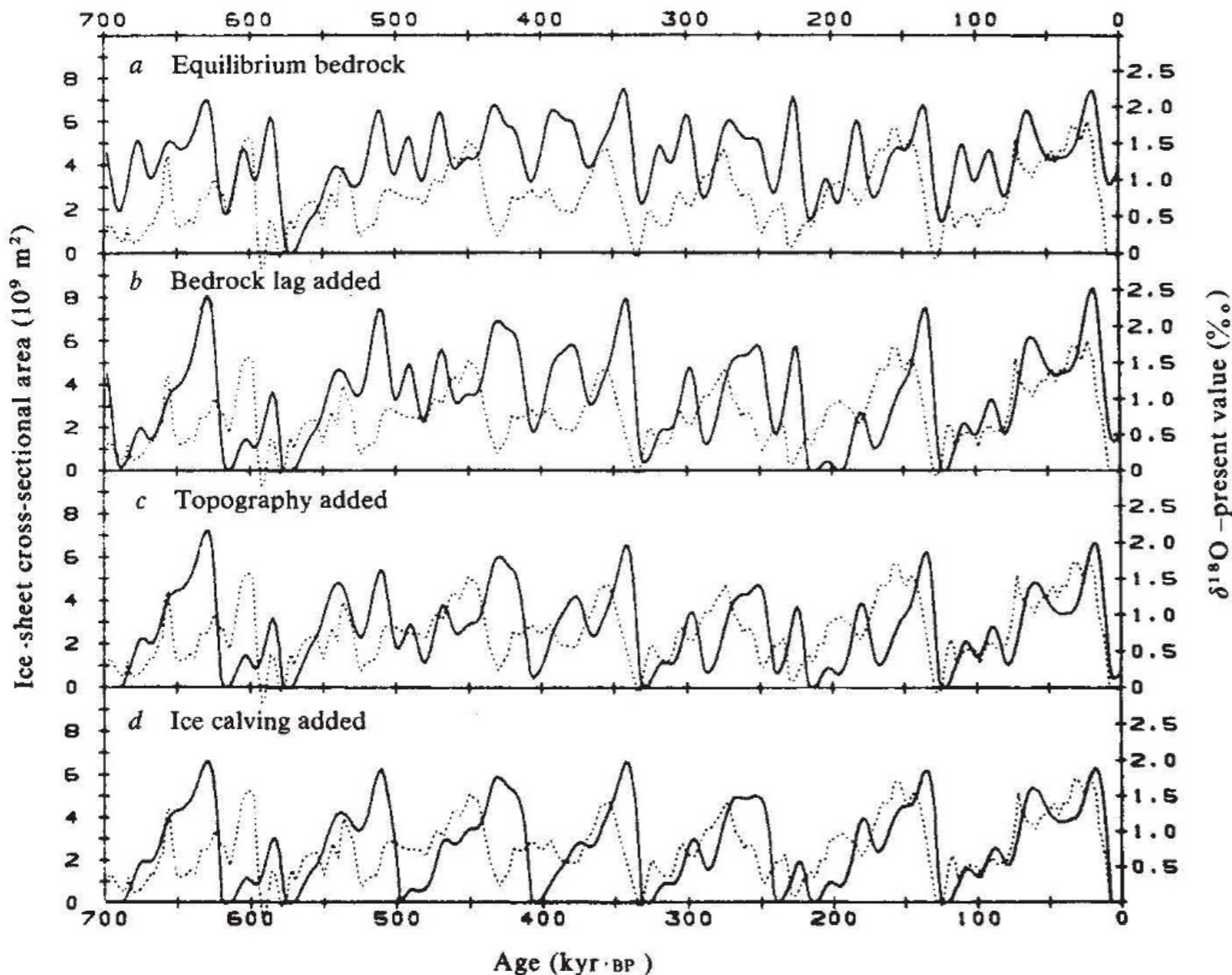
100 kyr cycle as a linear resonance, response to galactic dust clouds

100 kyr cycle from orbitally-forced ice sheet models

$$\frac{\partial h}{\partial t} = A \frac{\partial}{\partial x} \left[h^\alpha \left| \frac{\partial(h+h')}{\partial x} \right|^\beta \frac{\partial(h+h')}{\partial x} \right] + G(h+h', x, \text{orbit})$$

Shallow ice approximation with a prescribed surface mass balance

$$G = \begin{cases} a(h+h'-E) - b(h+h'-E)^2 & \text{if } h+h'-E \leq 1,500 \text{ m} \\ 0.56 & \text{if } h+h'-E > 1,500 \text{ m} \end{cases} \text{ m yr}^{-1}$$



A simple ice sheet model yields realistic 100 kyr glacial cycles

David Pollard, 1982

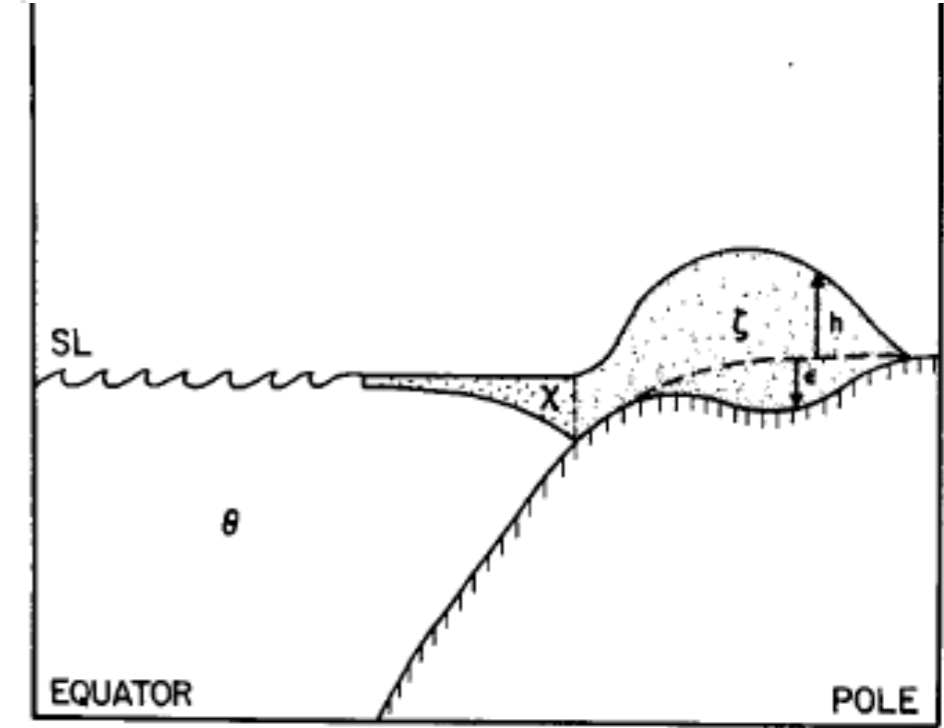
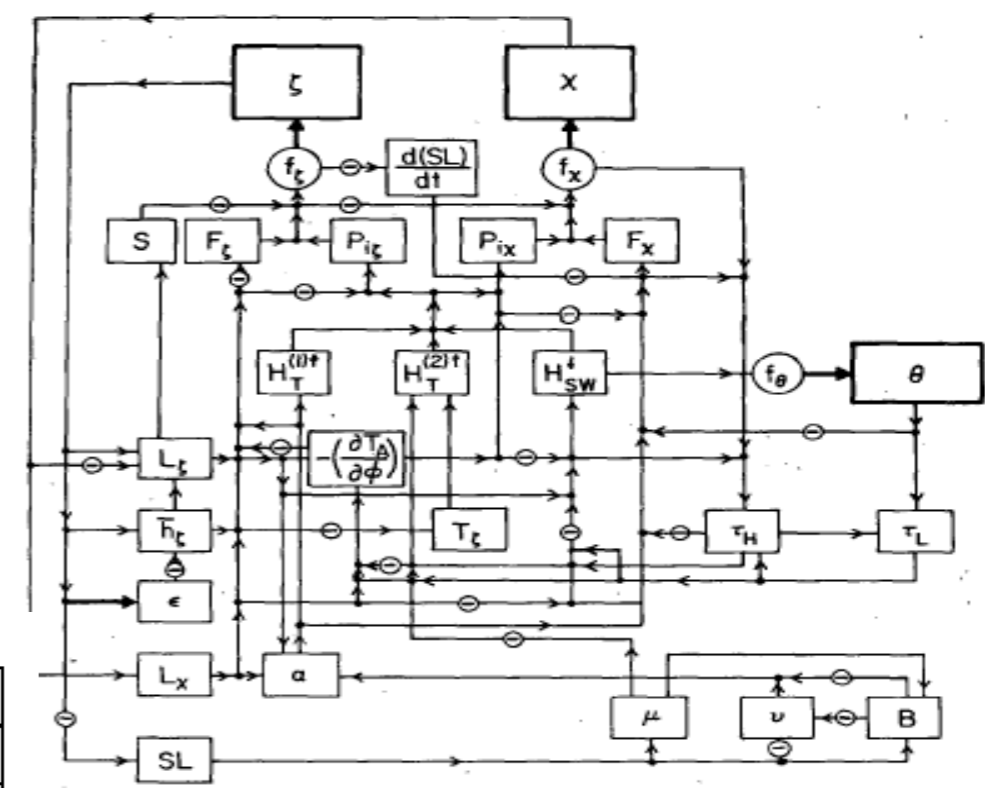
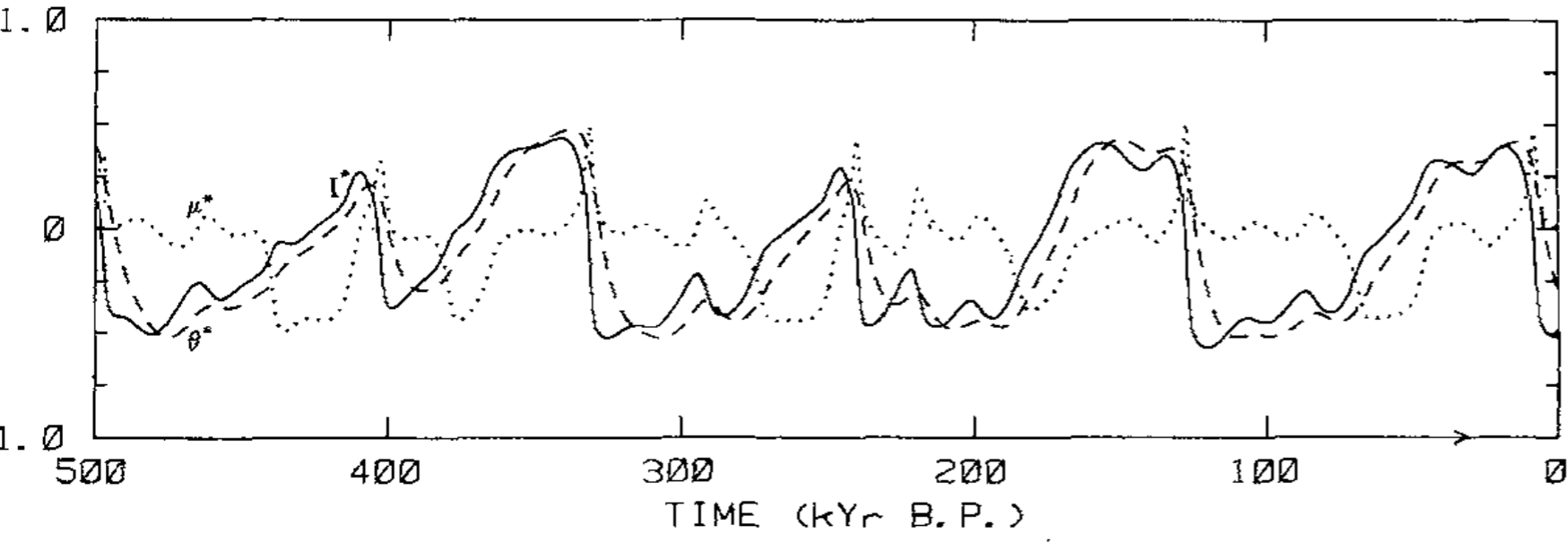
100 kyr cycle from toy "Earth system" models

Barry Saltzman: Carbon dioxide and the delta18O record of late-Quaternary climatic change: a global model (Clim Dyn, 1:77-85, 1987)

$$\frac{dX}{dt^*} = -\alpha_1 Y - \alpha_2 Z - \alpha_3 Y^2 \quad (17)$$

$$\frac{dY}{dt^*} = -\beta_0 X + \beta_1 Y + \beta_2 Z - (X^2 + 0.004 Y^2) Y + F_Y \quad (18)$$

$$\frac{dZ}{dt^*} = X - \gamma_2 Z \quad (19)$$

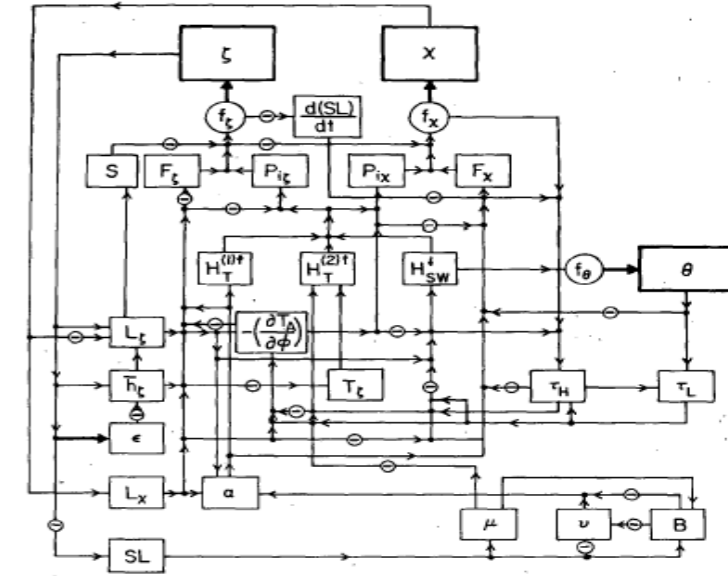
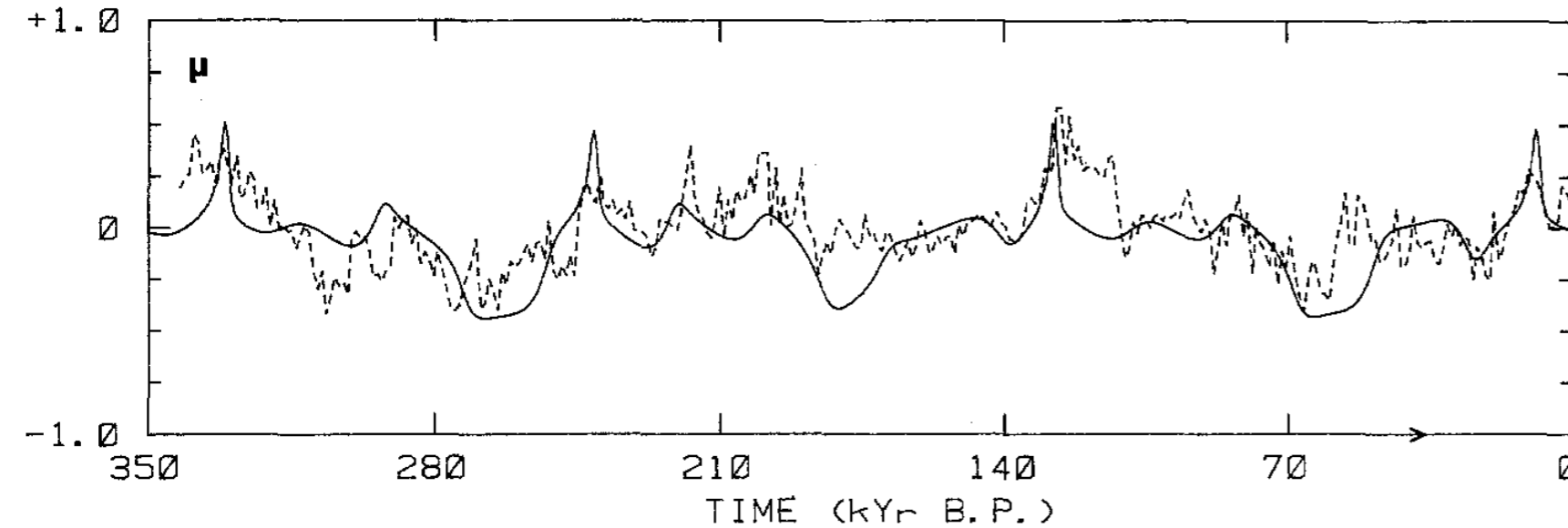


Orbitally-forced solution of the three-component dynamical system (17)-(19) in non-dimensional units, mu (solid curve), I* (dotted), and theta* (dashed)

100 kyr cycle from toy "Earth system" models

Barry Saltzman: Carbon dioxide and the delta18O record of late-Quaternary climatic change: a global model (Clim Dyn, 1:77–85, 1987)

A great fit to observations can be overdone....:

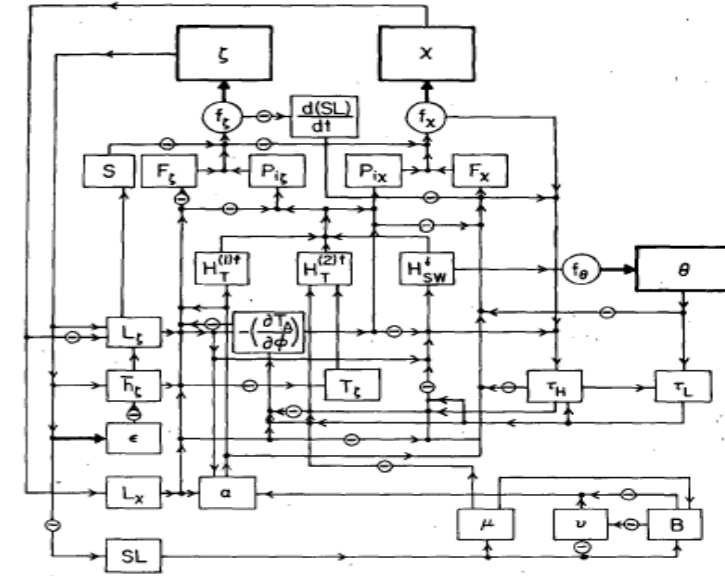
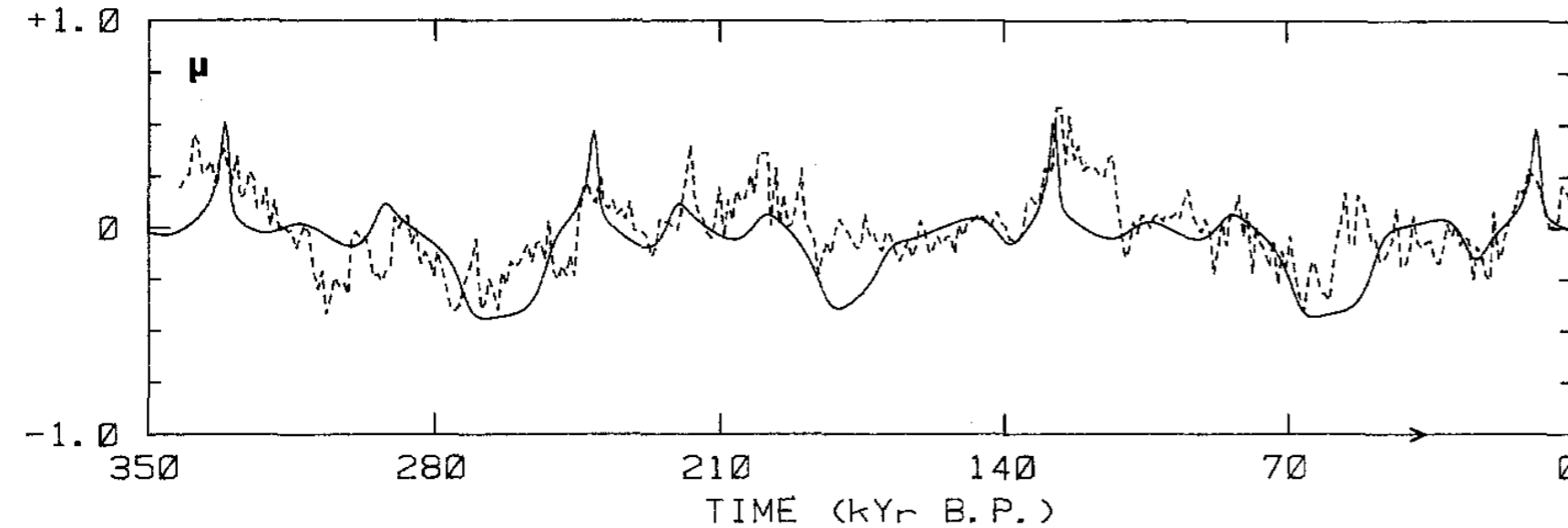


Comparison of solution for atmospheric carbon dioxide concentration (solid) with the variations inferred by Shackleton and Pisias (1985) for core V19-30 (dashed).

100 kyr cycle from toy "Earth system" models

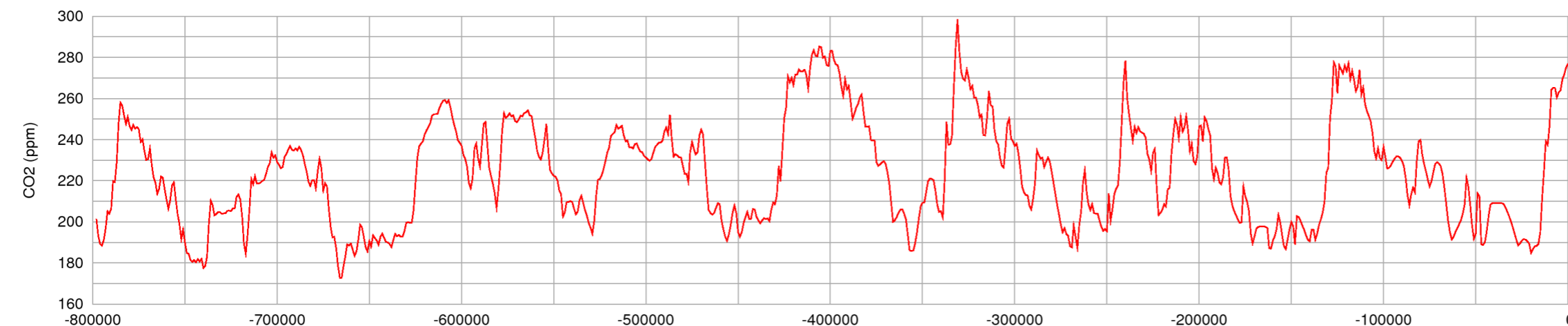
Barry Saltzman: Carbon dioxide and the delta18O record of late-Quaternary climatic change: a global model (Clim Dyn, 1:77–85, 1987)

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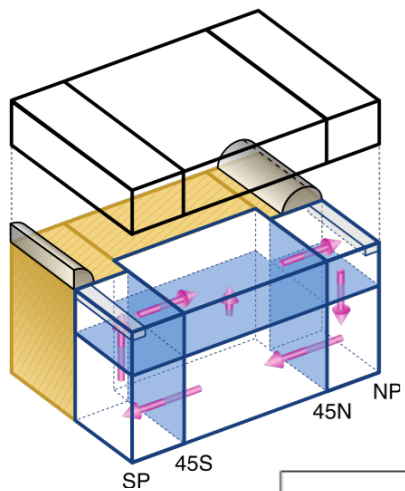


Comparison of solution for atmospheric carbon dioxide concentration (solid) with the variations inferred by Shackleton and Pisias (1985) for core V19-30 (dashed).

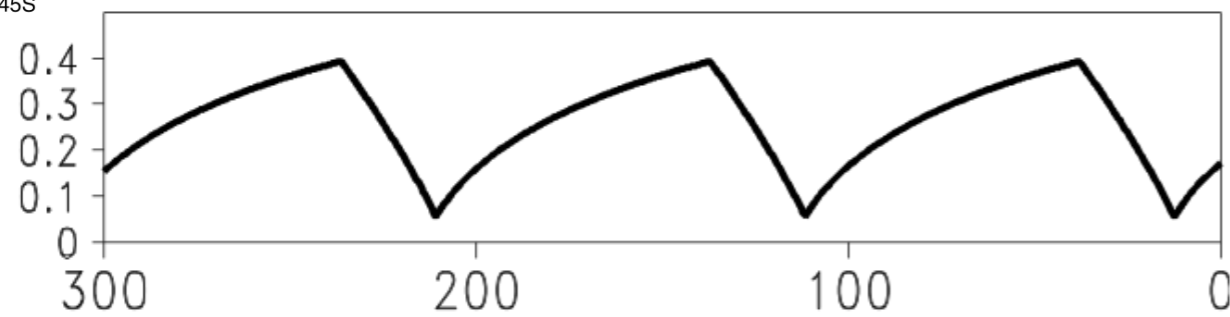
Unfortunately, the ice core record that came out soon after that, looks very different:



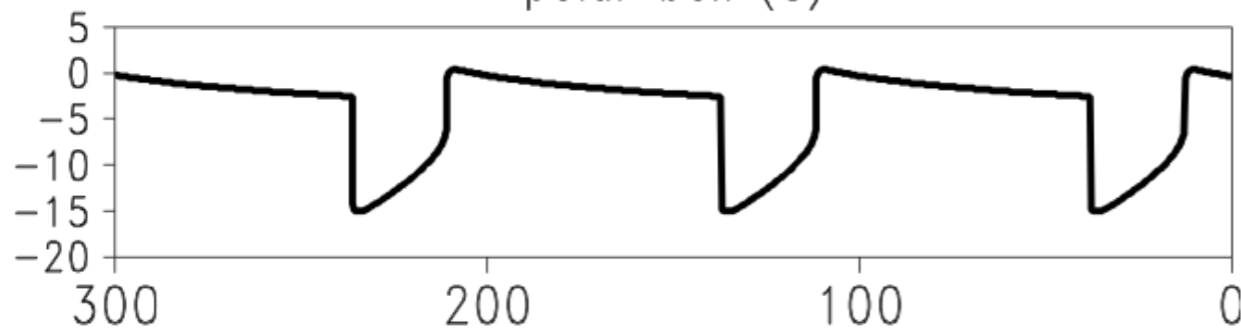
“Sea ice switch” glacial mechanism



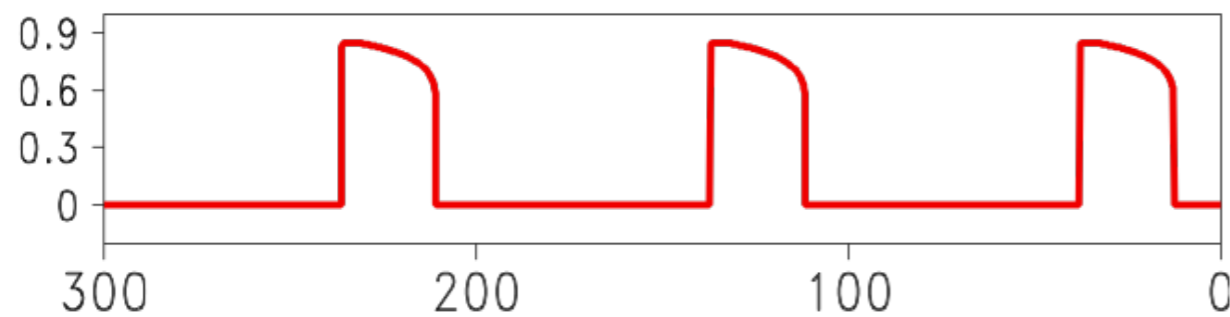
Meridional extent of land ice
in the Northern polar box



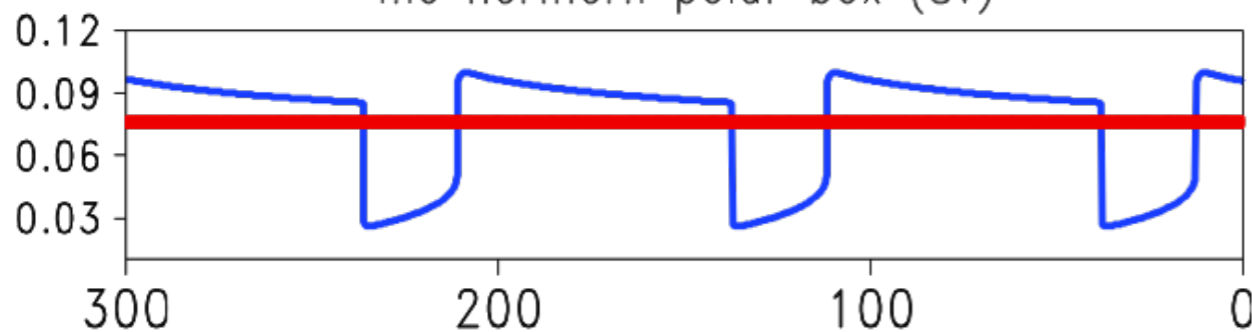
Atmospheric temperature in the northern
polar box (C)



Meridional extent of sea ice
in the Northern polar box

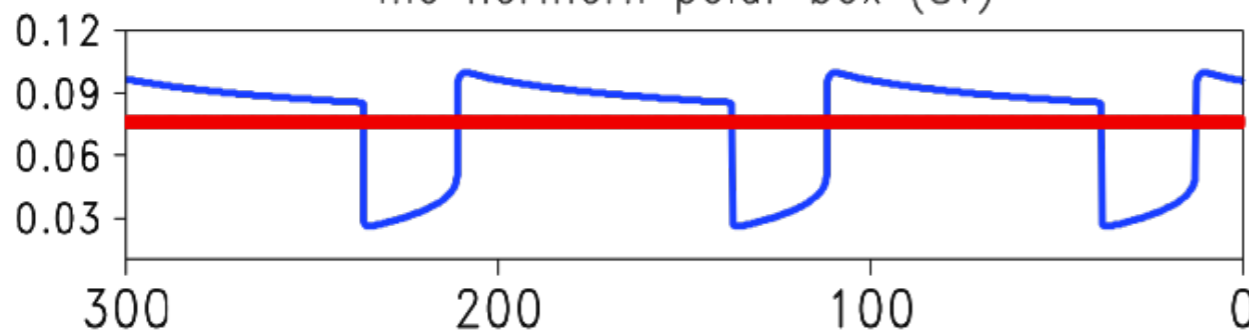
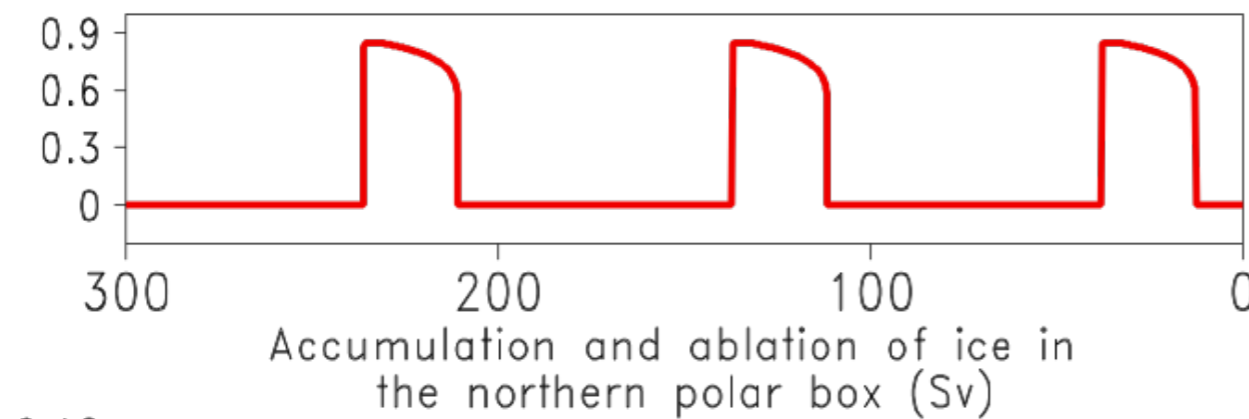
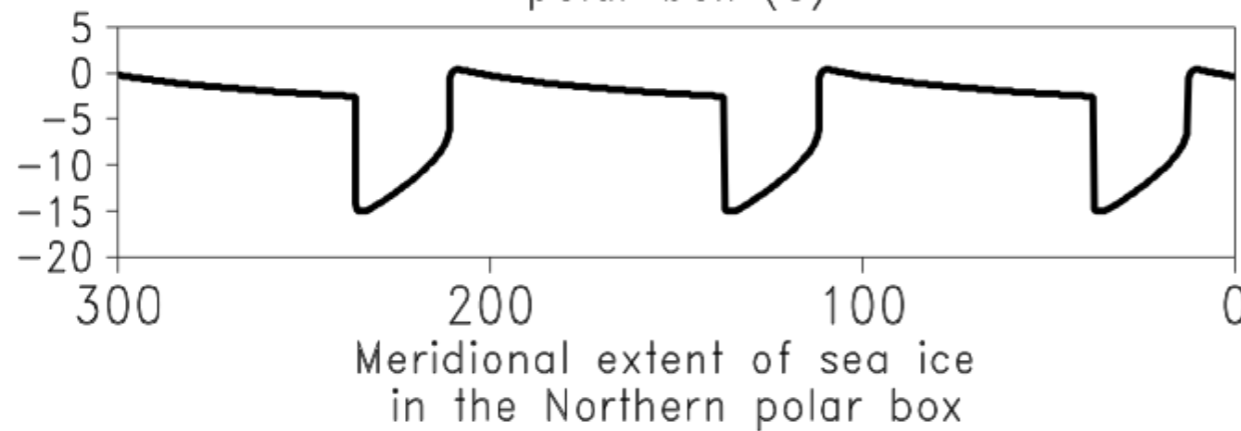
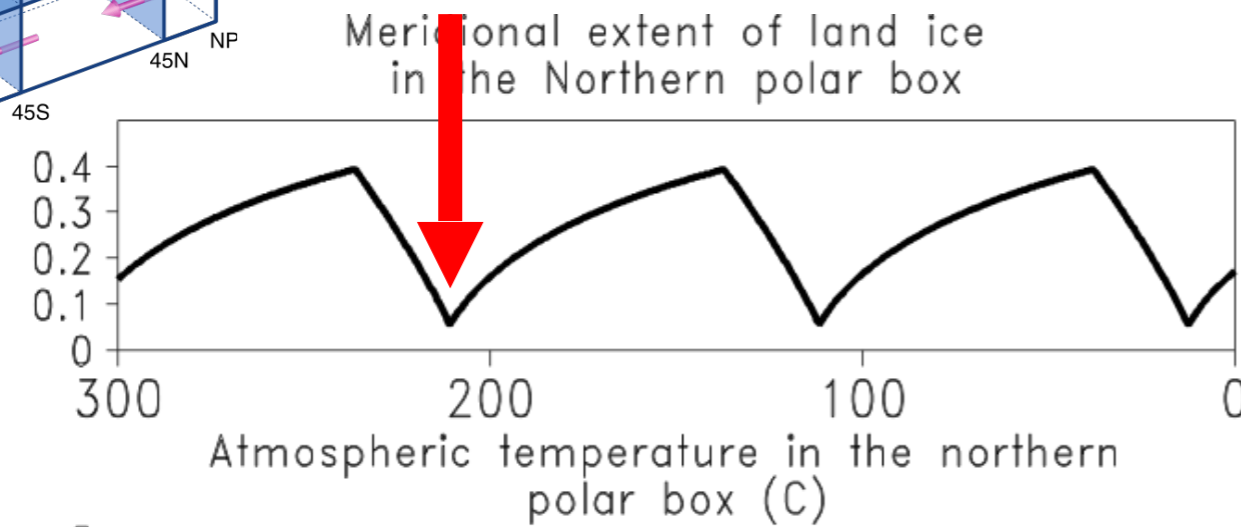
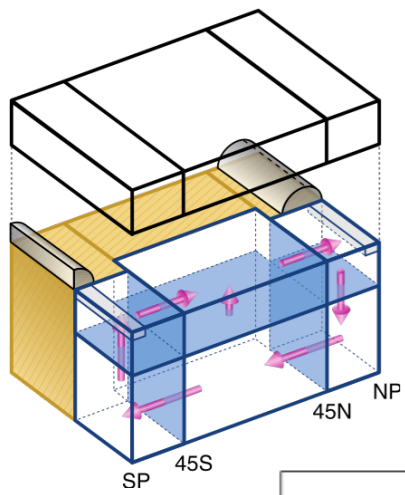


Accumulation and ablation of ice in
the northern polar box (Sv)

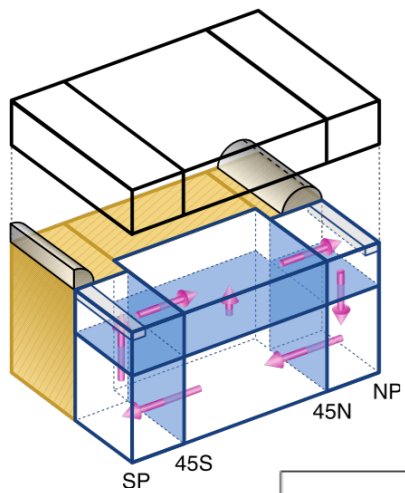


“Sea ice switch” glacial mechanism

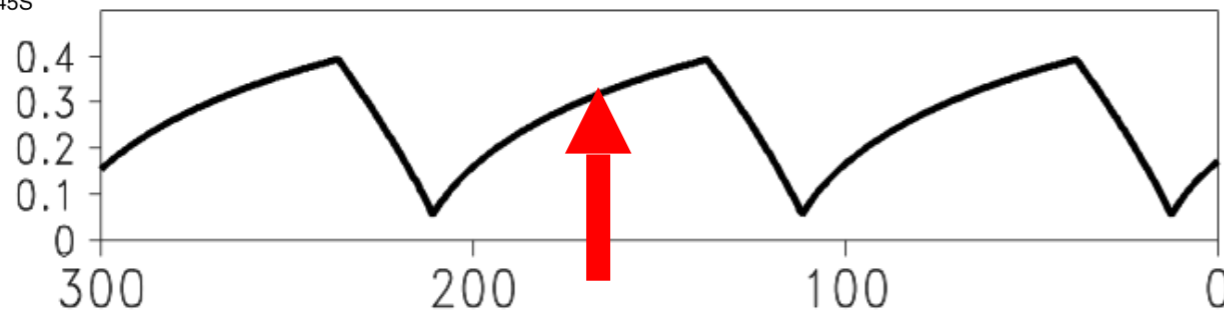
Interglacial, no ice, warm atmosphere, large accumulation.



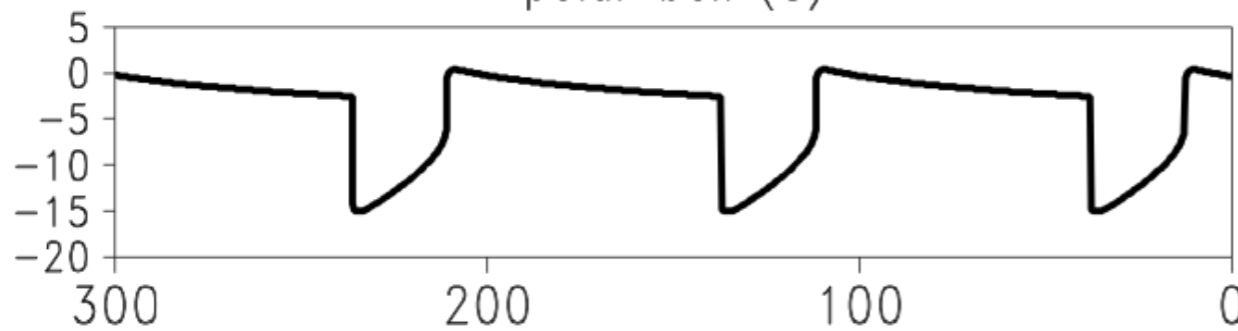
“Sea ice switch” glacial mechanism



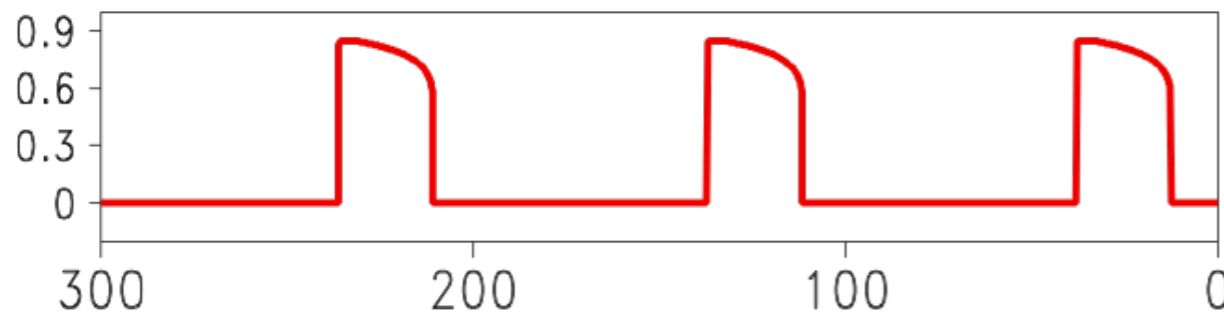
Meridional extent of land ice
in the Northern polar box



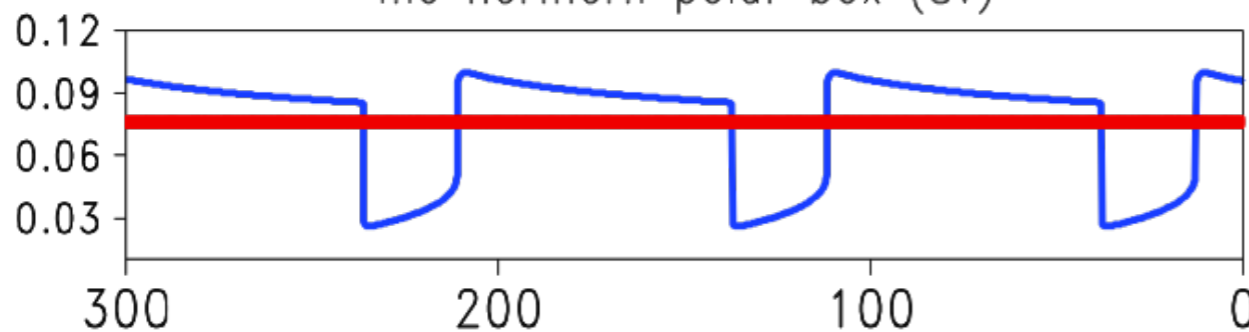
Atmospheric temperature in the northern
polar box (C)



Meridional extent of sea ice
in the Northern polar box



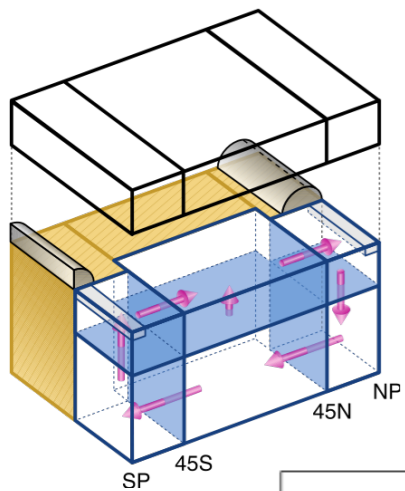
Accumulation and ablation of ice in
the northern polar box (Sv)



Interglacial, no ice, warm
atmosphere, large accumulation.

Land ice sheets grow, temperature
decreases.

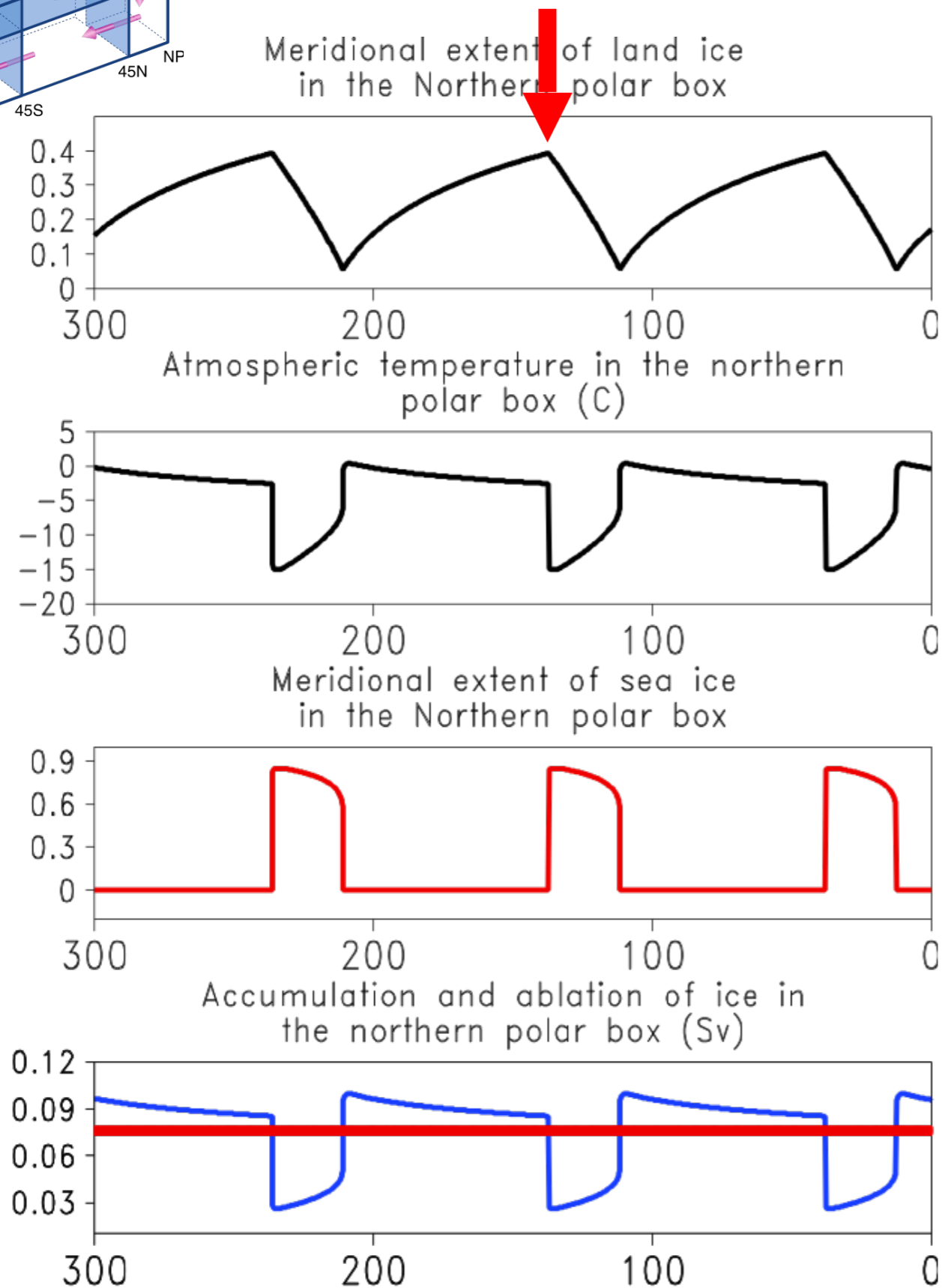
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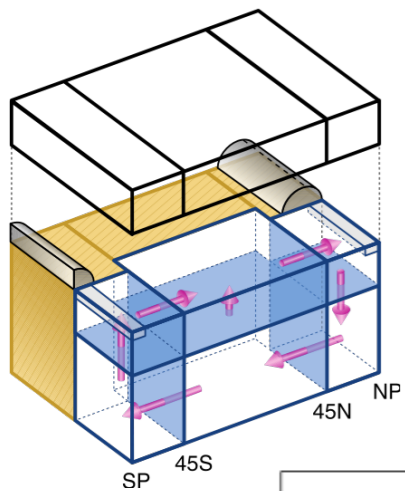
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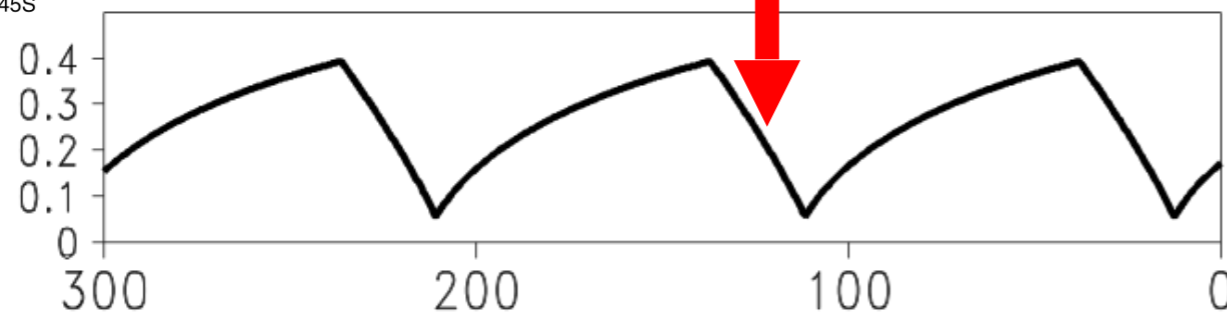
Cooling leads to sea ice formation, ice albedo feedback causes switch-like rapid sea ice growth.



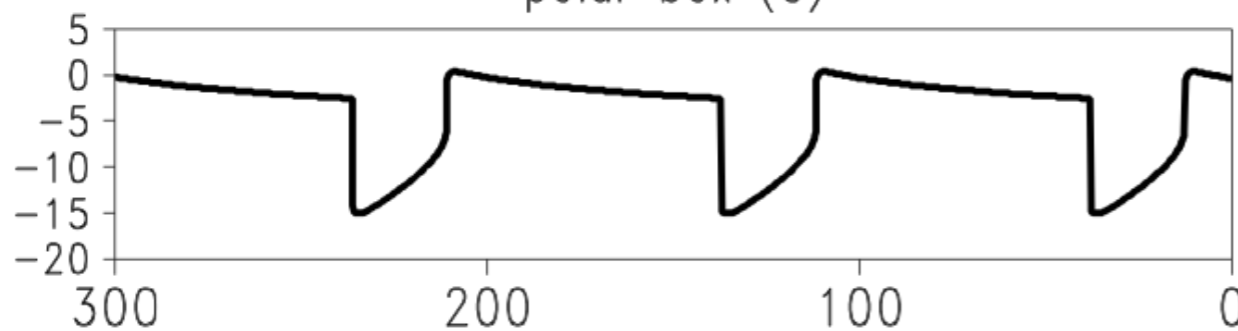
“Sea ice switch” glacial mechanism



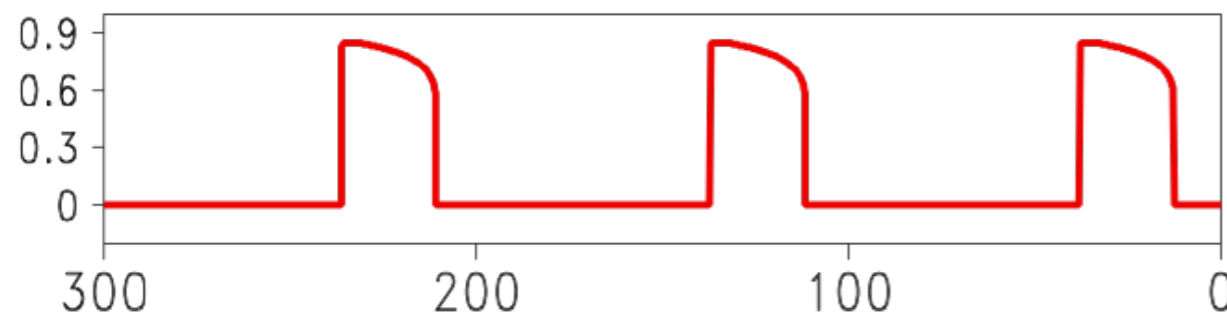
Meridional extent of land ice
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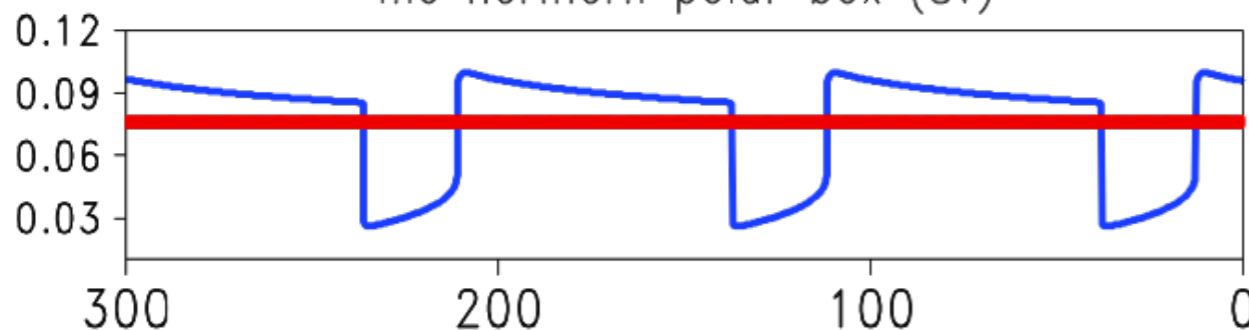
Atmospheric temperature in the northern
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Meridional extent of sea ice
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Accumulation and ablation of ice in
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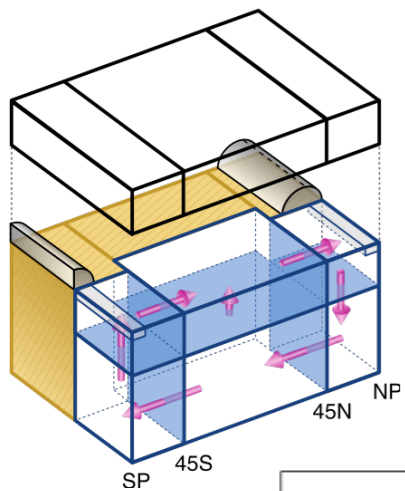
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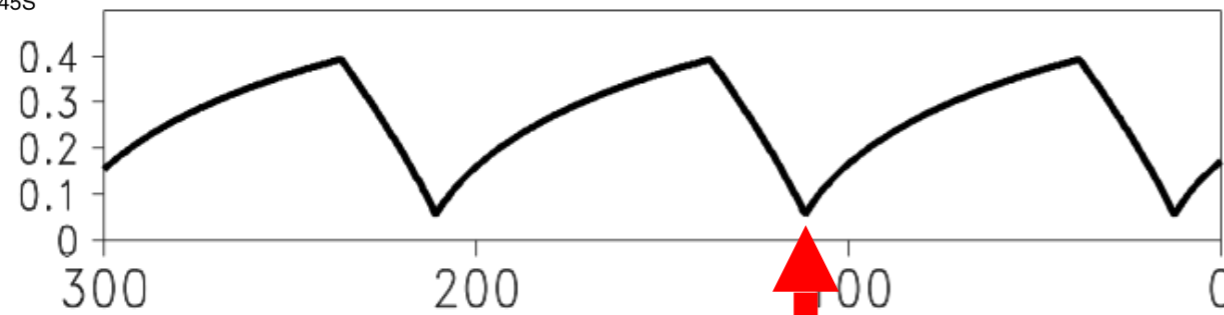
Cooling leads to sea ice formation, ice albedo feedback causes switch-like rapid sea ice growth.

Cold, dry, **weak accumulation**, **ablation continues**. Land ice retreats.

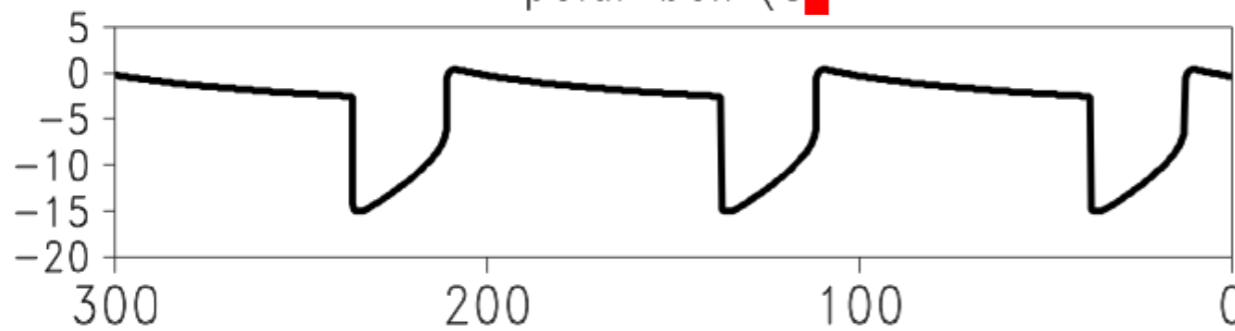
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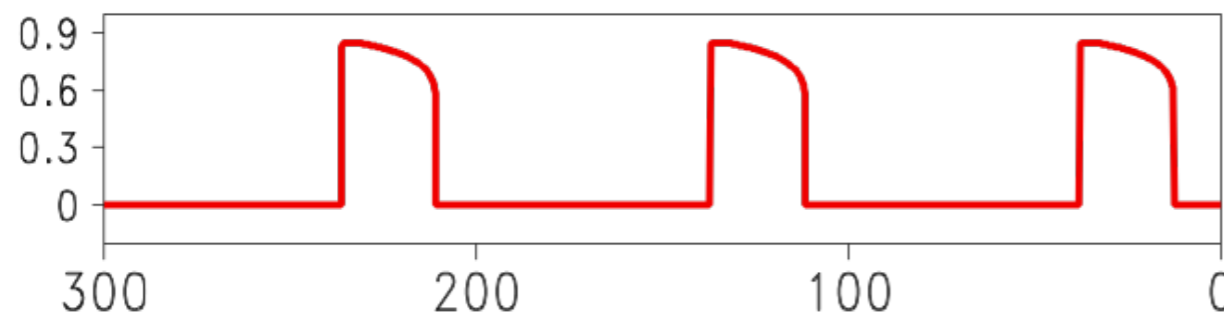
Meridional extent of land ice in the Northern polar box



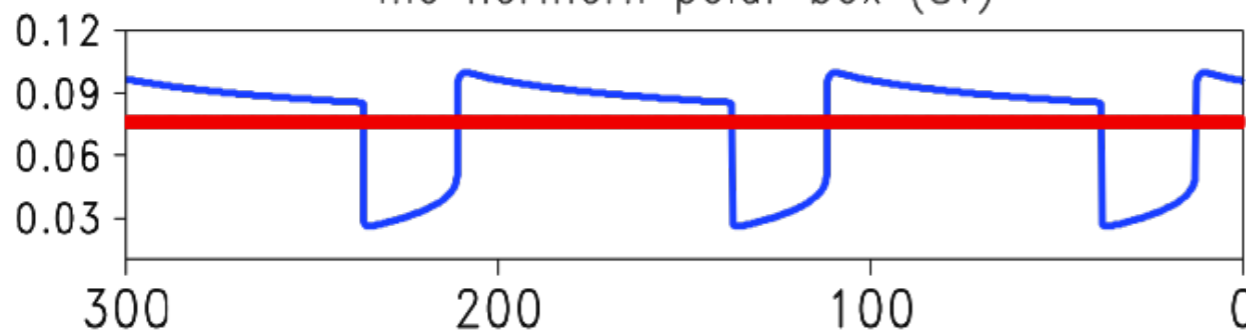
Atmospheric temperature in the northern polar box (C)



Meridional extent of sea ice in the Northern polar box



Accumulation and ablation of ice in the northern polar box (Sv)



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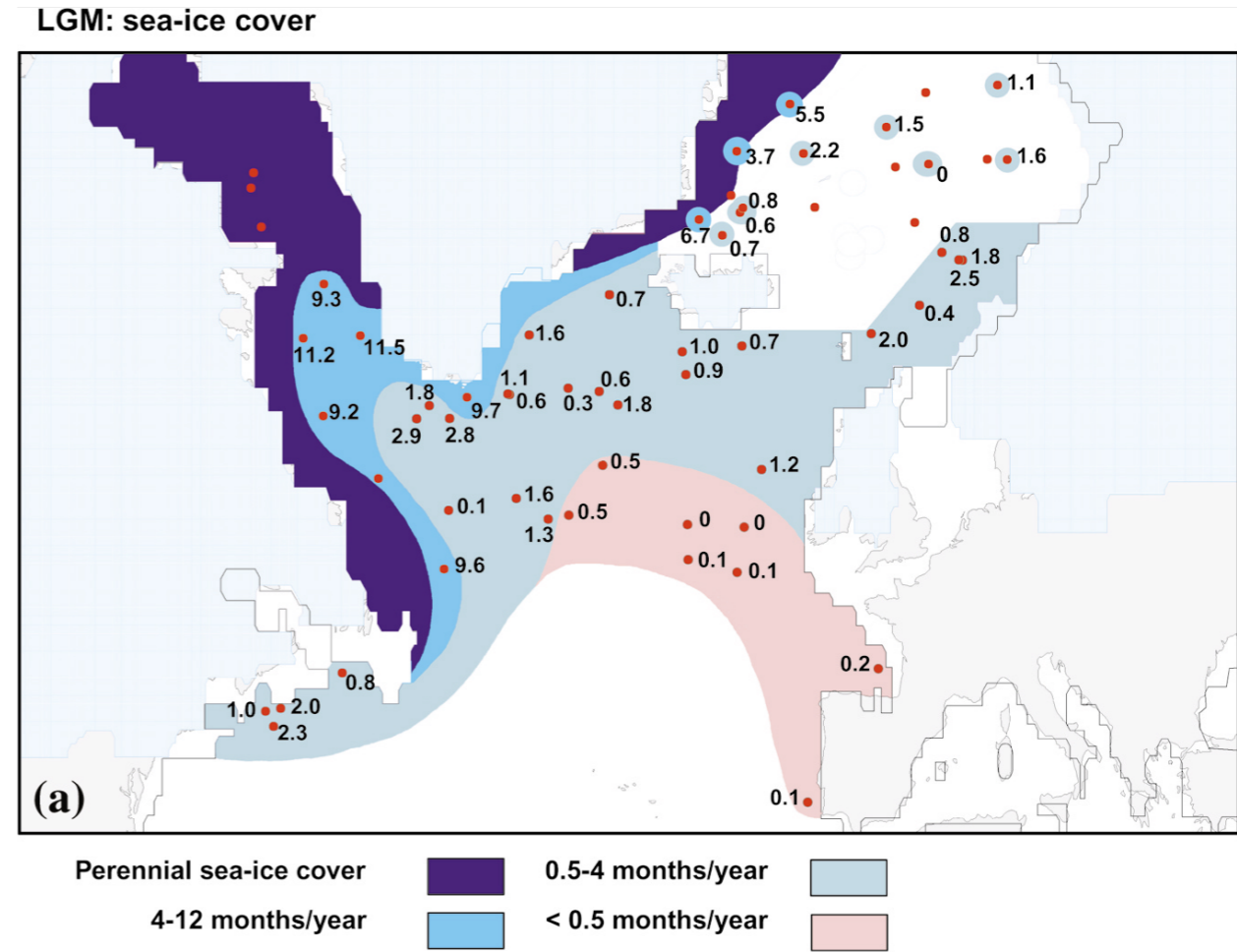
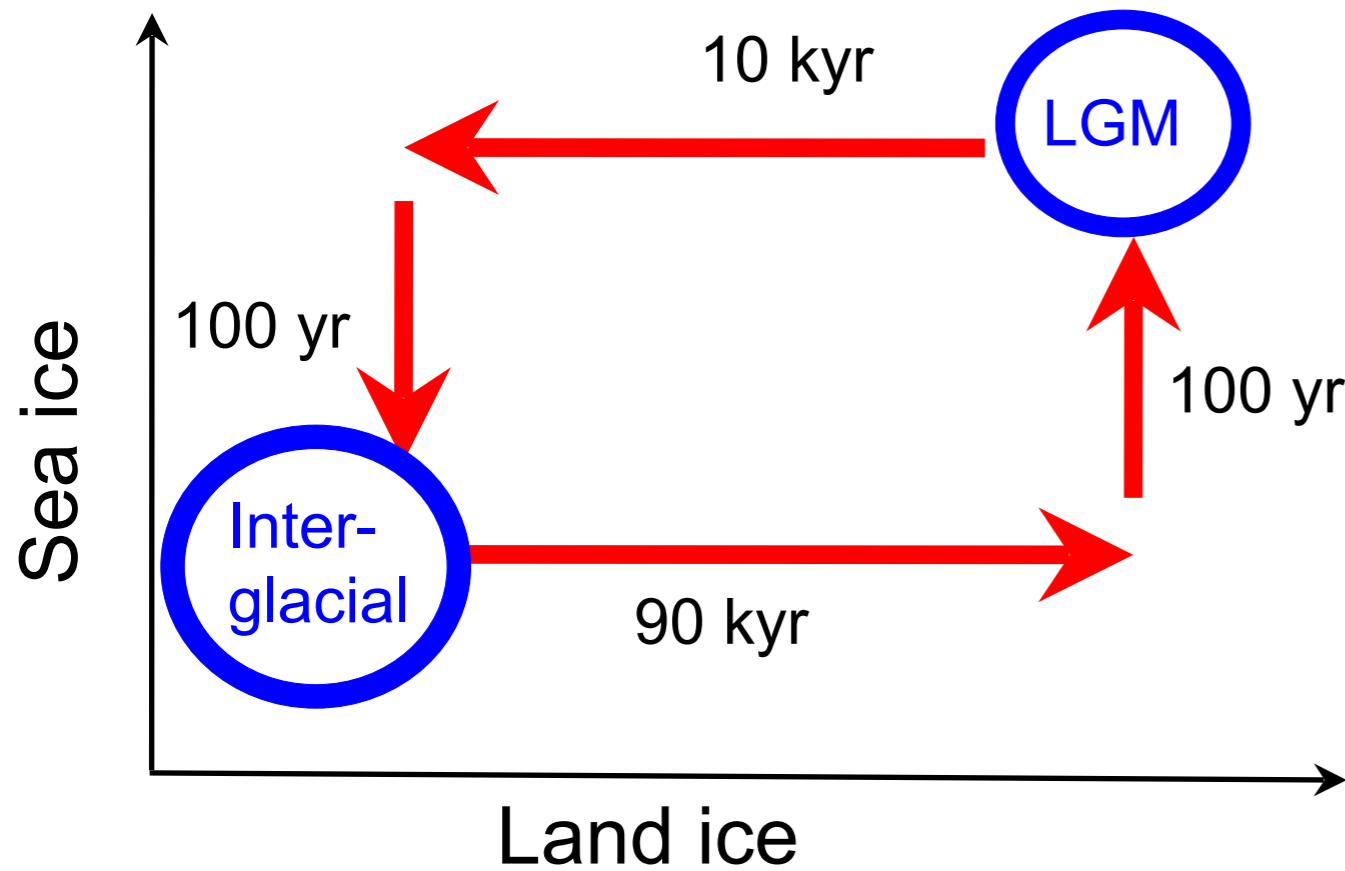
Sea ice melts, first slowly then again abruptly due to sea ice feedbacks. Back to initial state.

Sea ice 'pancakes'



Observational evidence for sea ice switch?

Mechanism predicts a hysteresis between sea ice & land ice



Currently, only sea ice proxy at a single time slice, LGM (21kyr) exists. (Anne de Vernal et al, 2000)

This mechanism, unlike many others, is falsifiable: need to examine phase between land ice & sea ice.

Insolation-driven glacial hysteresis

(Abe-Ouchi et al 2013)

A coupled ice-sheet climate model shows a hysteresis loop with insolation. When Insolation varies with Milankovitch forcing, this leads to glacial-like oscillations

This represents a case where there would have been no glacial oscillations without time-varying Milankovitch forcing.

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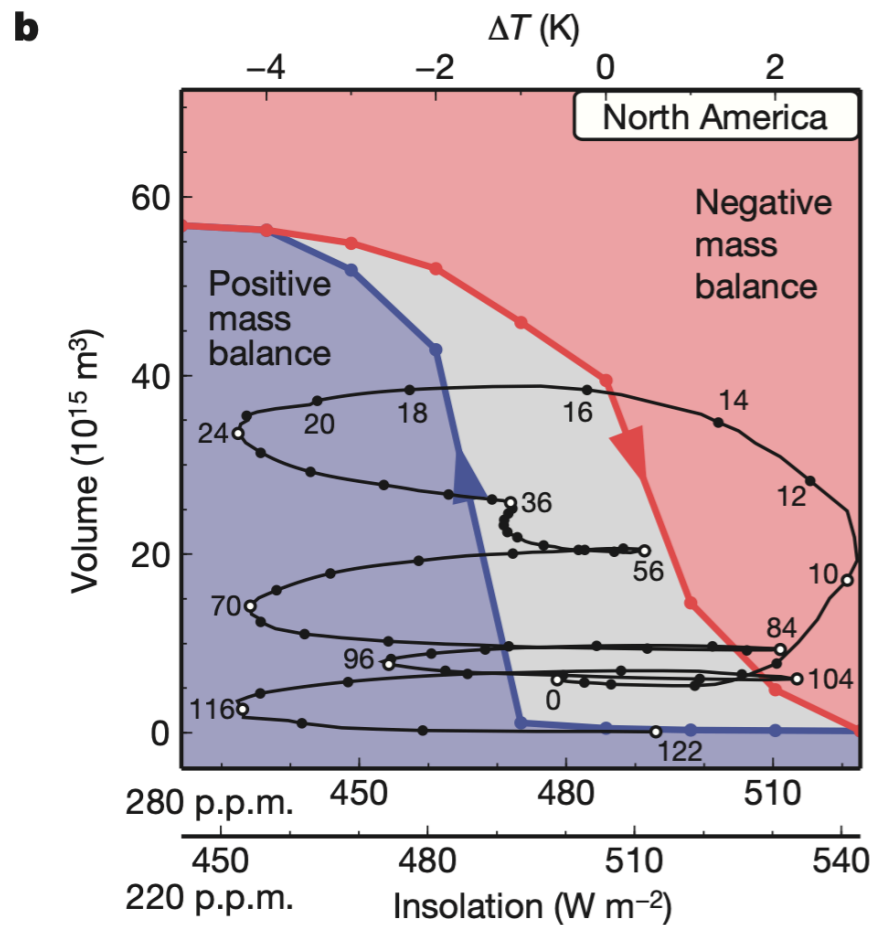
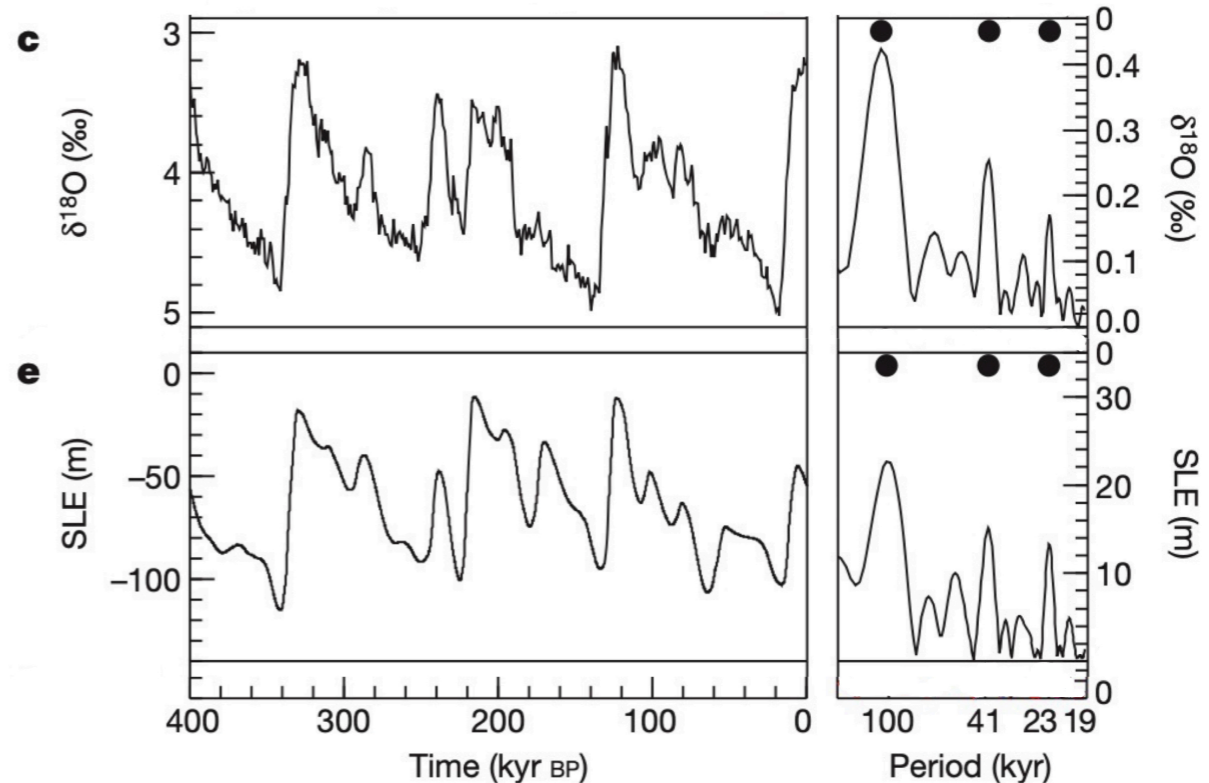


Fig 2b: Ice volume as a function of insolation. Showing hysteresis loop (upper branch in red, lower in blue), and model trajectory (black) for last 122 kyr under time-dependent Milankovitch insolation.



Ice volume time series and spectrum for observations (1c) and model run for the last 400 kyrs (1e).

This represents a case where there would have been no glacial oscillations without time-varying Milankovitch forcing.

Milankovitch forcing:

Variations in the Earth's Orbit: Pacemaker of the Ice Ages

For 500,000 years, major climatic changes have followed variations in obliquity and precession.

J. D. Hays, John Imbrie, N. J. Shackleton 1977

“pacemaker” = nonlinear phase locking?

First: analysis of phase locking of fireflies

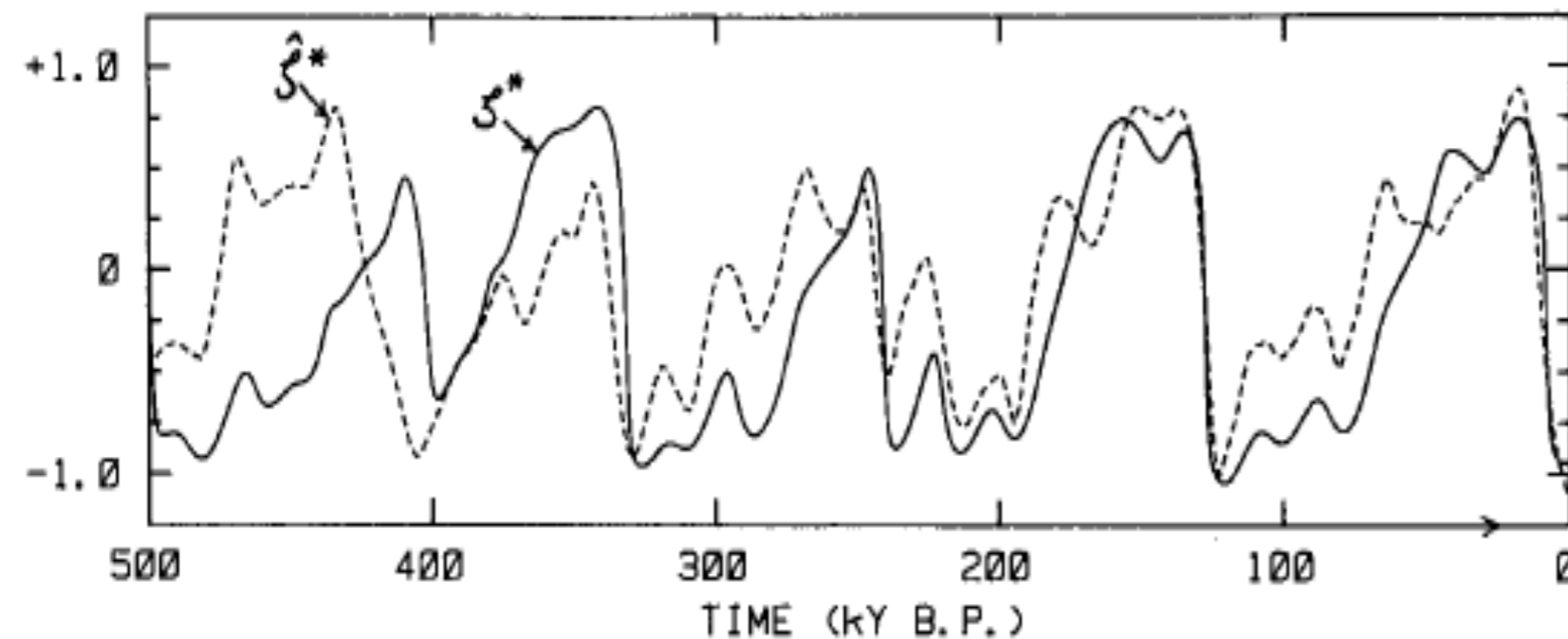
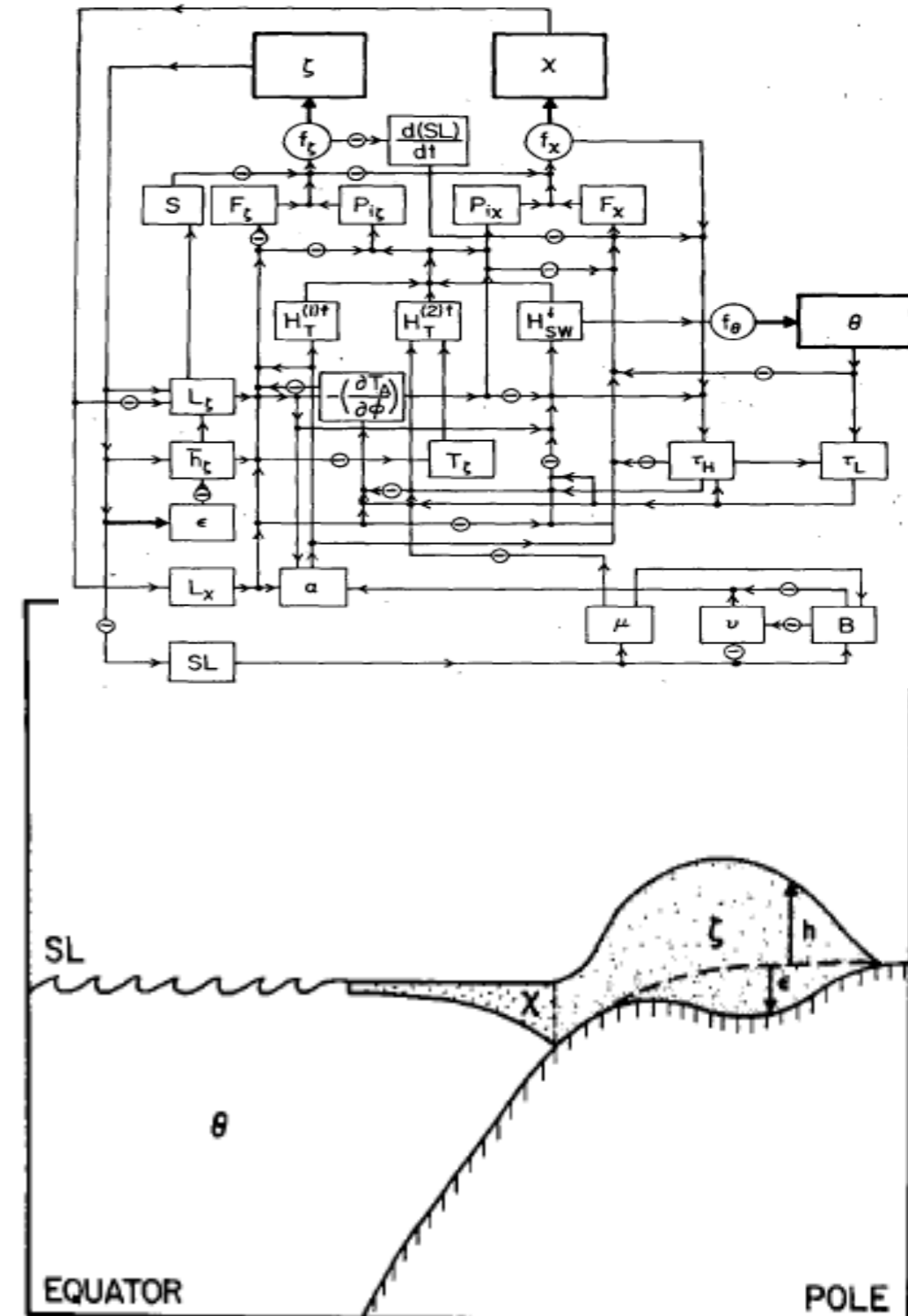
Three “successful” glacial models: 1

Saltzman, Hansen and Maasch, 1984: glacial cycles are due to the interaction of land ice, ice shelves and deep ocean temp.

$$\frac{d\zeta'}{dt} = a_0\zeta' + a_1(1 - a_2\chi')\chi' - a_3\theta' + F_\zeta + R_\zeta, \quad (1)$$

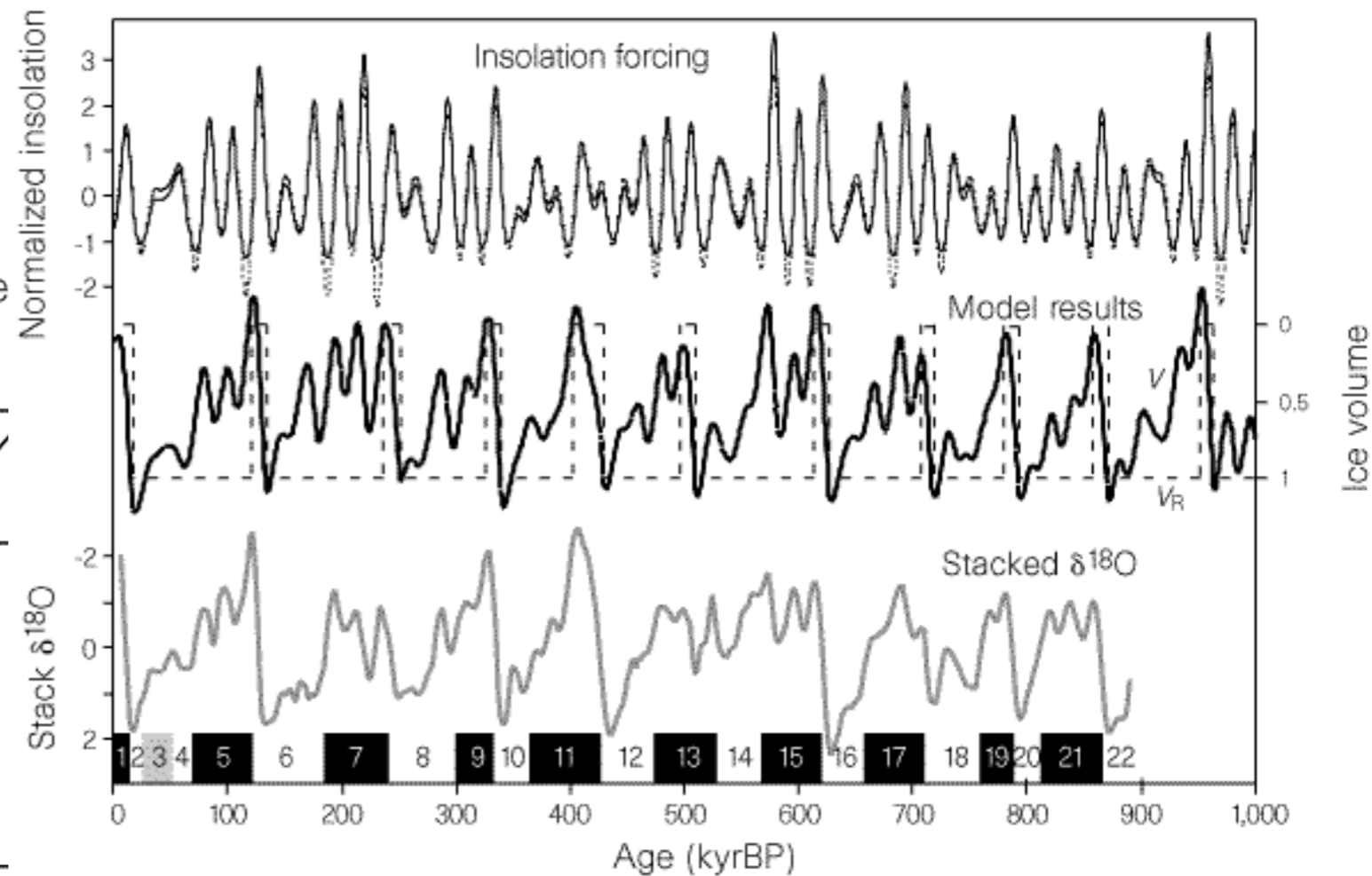
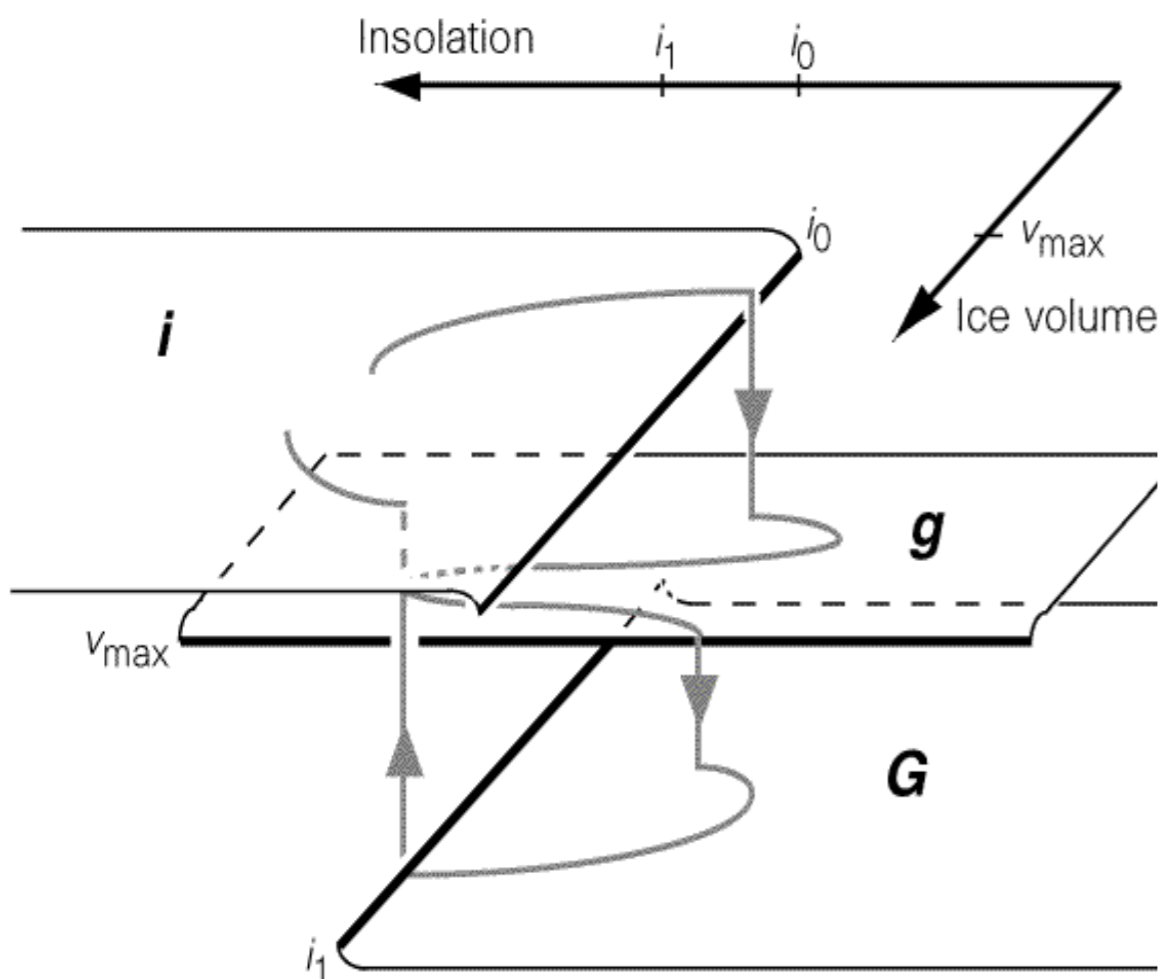
$$\frac{d\chi'}{dt} = b_0\zeta' + b_1(1 - b_2\chi' - b_3\chi'^2 - b_4\zeta'^2)\chi' - b_5\theta' + F_\chi + R_\chi, \quad (2)$$

$$\frac{d\theta'}{dt} = c_0\zeta' + c_1\chi' - c_2\theta' + F_\theta + R_\theta, \quad (3)$$



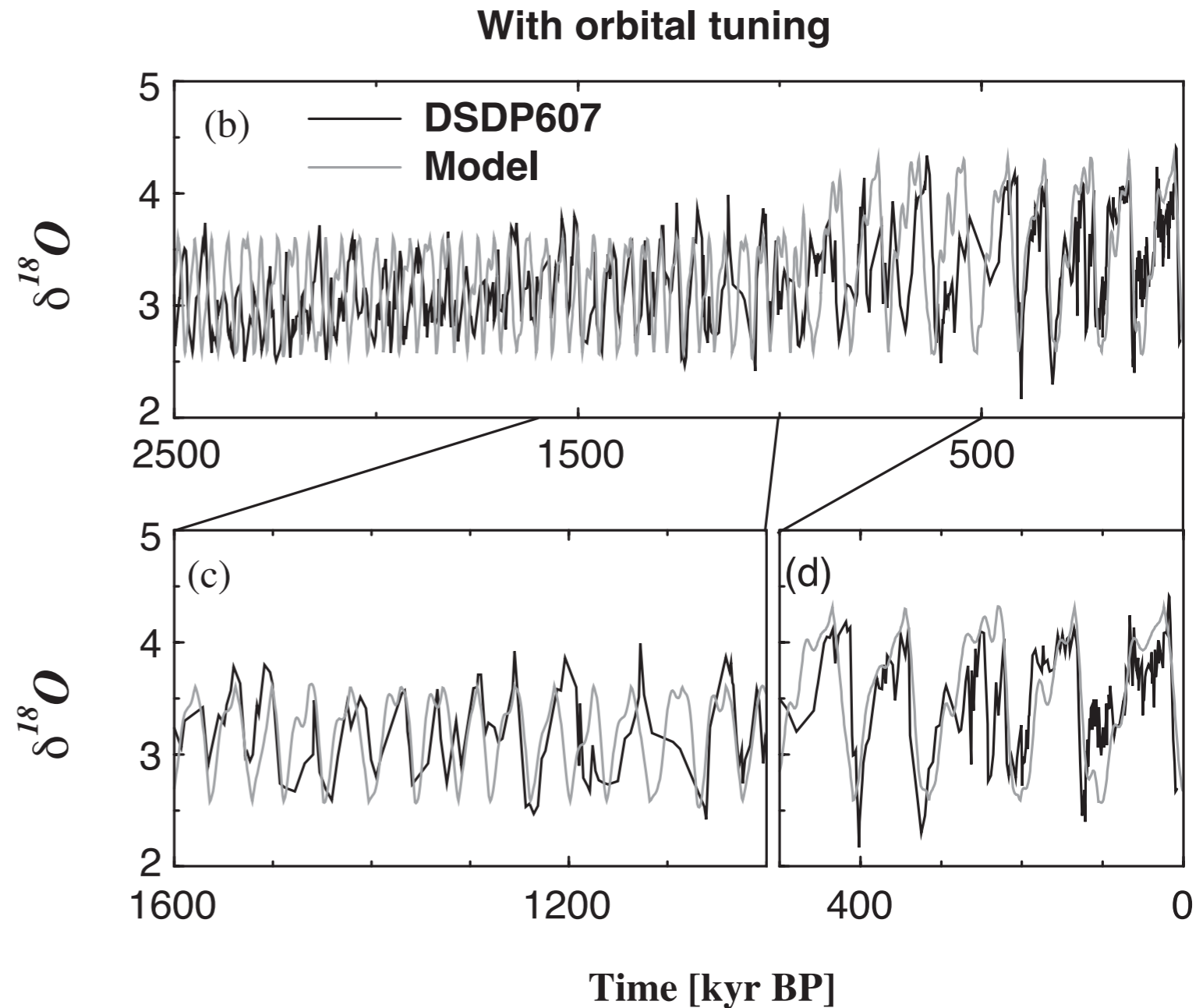
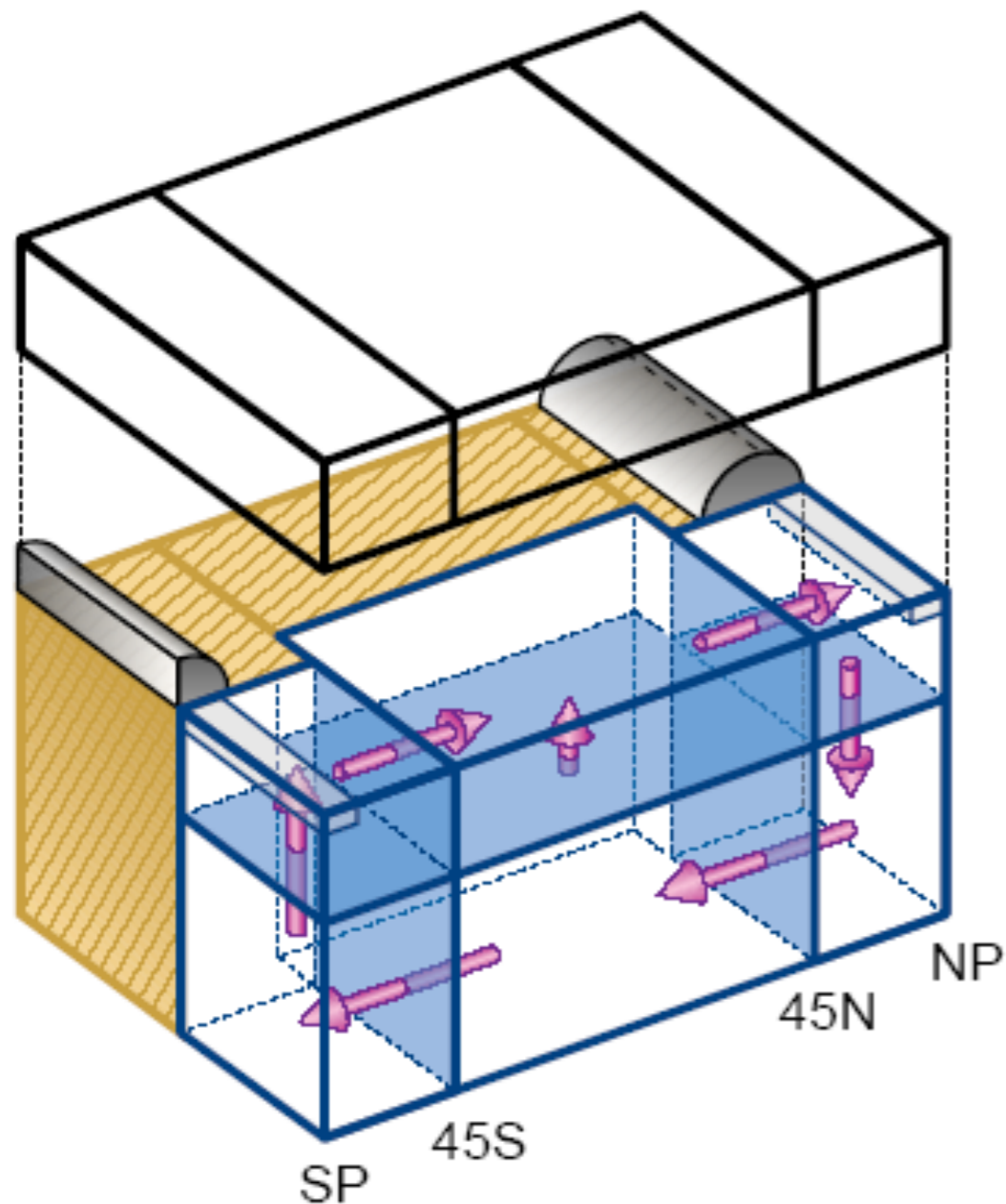
Three “successful” glacial models: 2

- Paillard 1998: 3 steady states, one equation, transition between steady states based on Milankovitch forcing.
- Glacial cycles are due to jumps between steady states (of THC?) forced by Milankovitch



Three “successful” glacial models: 3

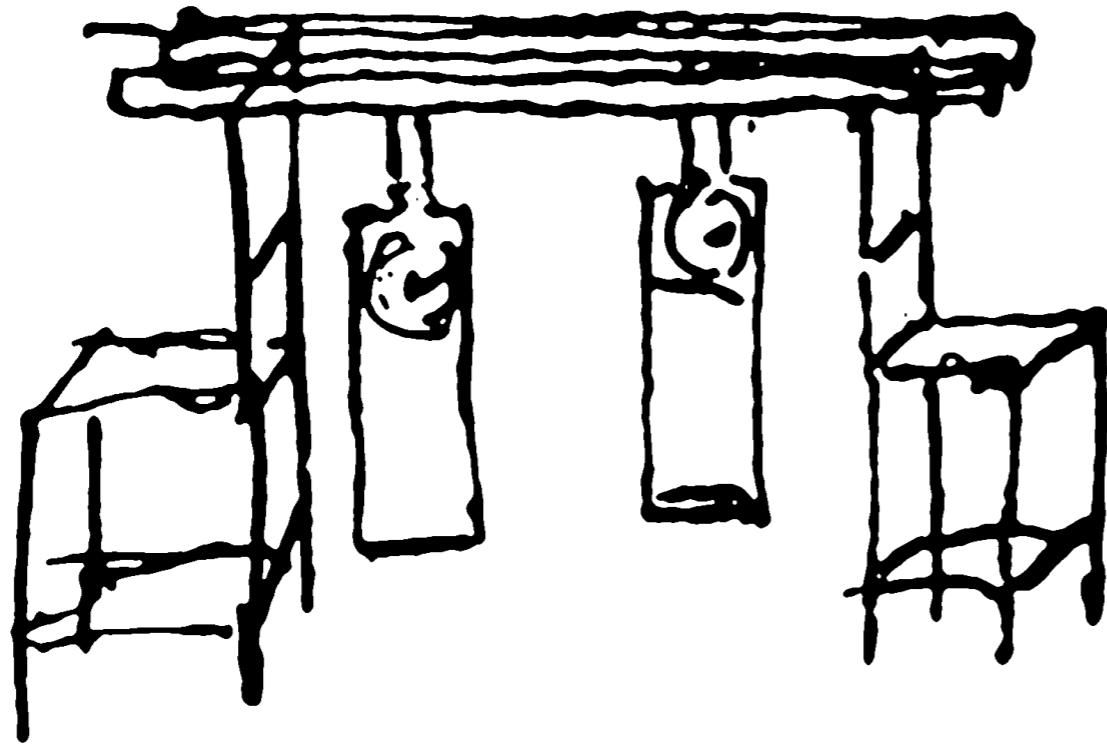
Gildor and Tziperman 2000: “sea ice switch”: land ice grows during warm periods (small sea ice cover) and retreats during cold periods (large sea ice cover);



(Ashkenazy & Tziperman 2004)

Why is it so 'simple' to fit glacial ice volume?

The question (Saltzman, Hansen & Maasch 1984): “How does small amplitude periodic forcing control phase in a complex nonlinear oscillatory system, and is there a good physical interpretation for this phase locking phenomenon?”



The answer: It's “nonlinear phase locking”: (Hyde&Peltier 1987; Gildor&Tziperman 2000; Ashkenazy&Tziperman 2004)

1665, Christiaan Huygens, Dutch mathematician, astronomer and physicist. While working on design of precise pendulum clocks, suitable for determination of a ship coordinates in the sea, he observed and described synchronization of two clocks placed on a common support.

<http://www.agnld.uni-potsdam.de/~mros/synchro.html>

Examples of nonlinear phase locking: 1/3

Moon always faces Earth: makes one rotation around its axis for every rotation around Earth.



NASA,
<https://www.cronodon.com/PlanetTech/mercury.html>



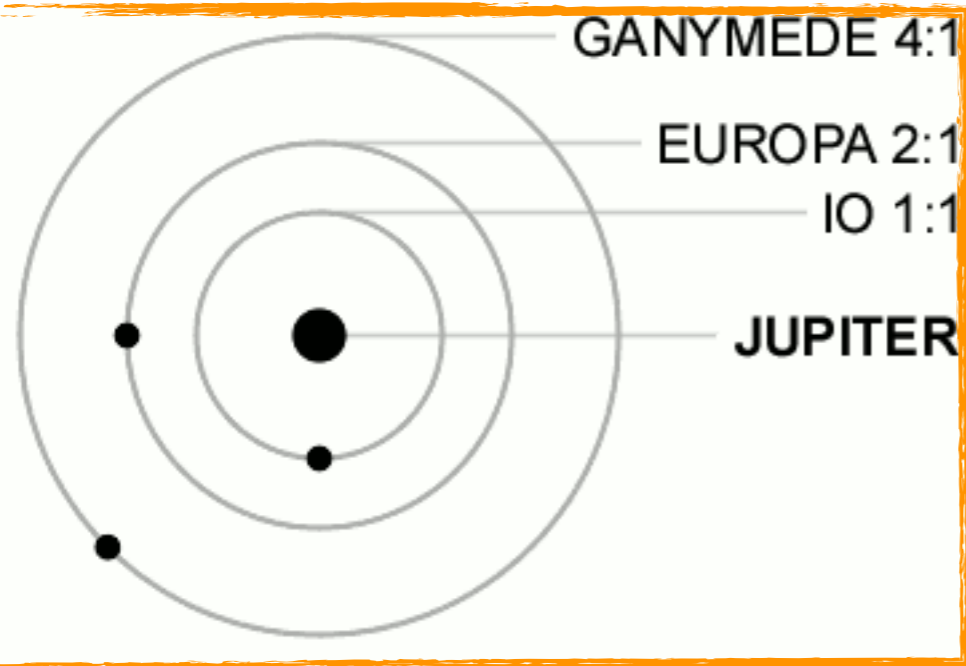
<https://www.britannica.com/list/where-did-the-moon-come-from>
 © Paul Morley/Fotolia

The planet Mercury turns three times for every two orbits around the Sun, this is a 2:3 nonlinear resonance

- Coincidences? No... It's **nonlinear phase locking/ nonlinear resonance** aided by tidal friction.
- Linear resonance: 1:1 ratio between forcing and response;
- Nonlinear resonance: $p:q$ ratio with p, q any integers.

Examples of nonlinear phase locking: 1/3

The inner three Galilean moons of Jupiter (Io, Europa, and Ganymede) are locked in a 4:2:1 orbital resonance, also known as a Laplace resonance, where for every four orbits of Io, Europa completes two, and Ganymede completes one. https://en.wikipedia.org/wiki/Orbital_resonance; <https://www.system-sounds.com/jupiters-moons/>



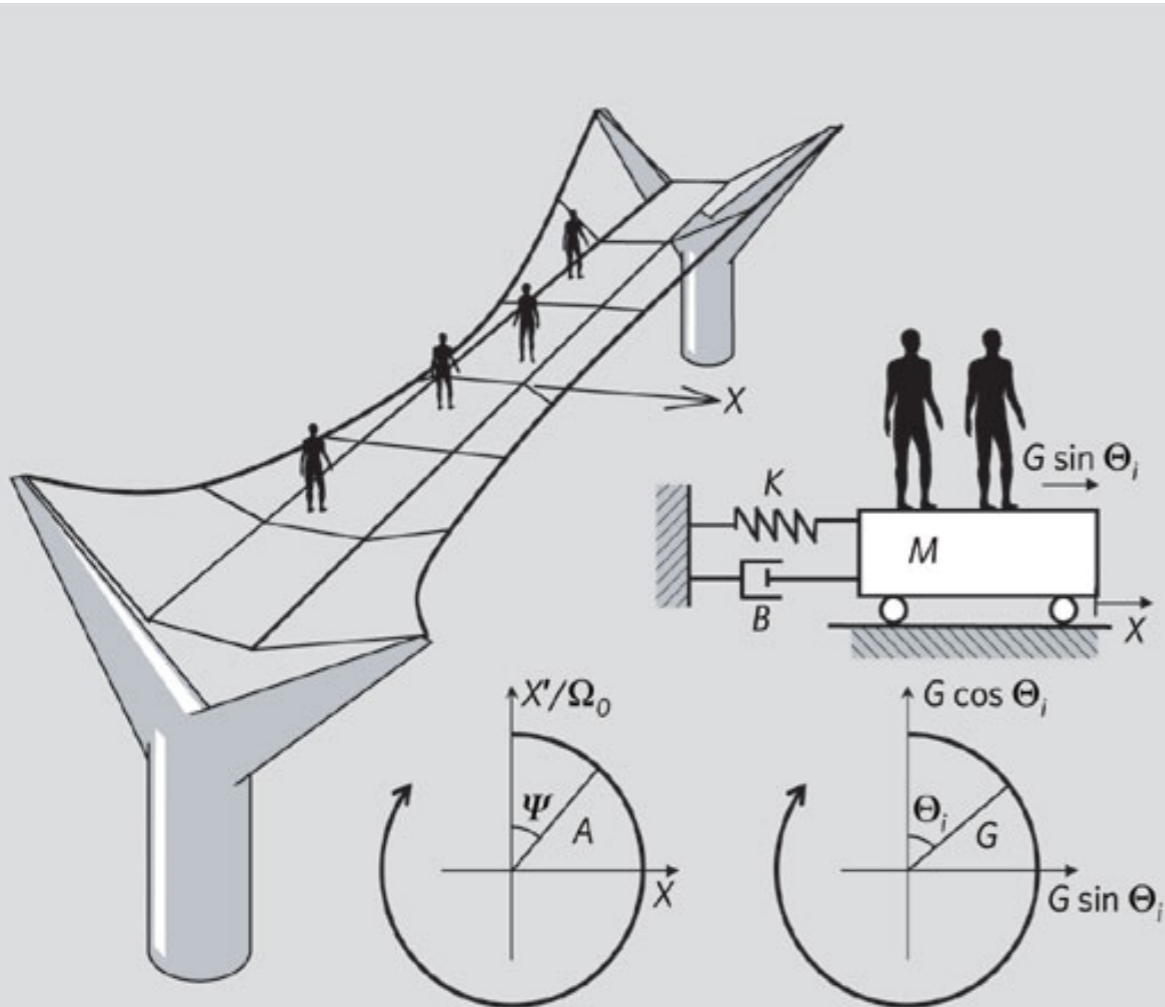
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- Nonlinear resonance: $p:q$ ratio with p, q any integers.

Examples of nonlinear phase locking: 2/3

Crowd synchrony on the Millennium Bridge

Footbridges start to sway when packed with pedestrians falling into step with their vibrations. (Strogatz et al 2005)

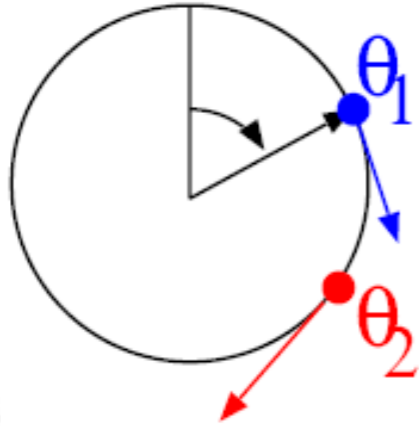


$$M \frac{d^2 X}{dt^2} + B \frac{dX}{dt} + KX = G \sum_{i=1}^N \sin \Theta_i$$

$$\frac{d\Theta_i}{dt} = \Omega_i + CA \sin(\Psi - \Theta_i + \alpha)$$

Bridge
and pedestrians
affecting each other nonlinearly

Examples of nonlinear phase locking: 3/3



$$\dot{\theta}_1 = \omega_1 + K_1 \sin(\theta_2 - \theta_1)$$

$$\dot{\theta}_2 = \omega_2 + K_2 \sin(\theta_1 - \theta_2)$$

$\omega_1 = \omega_2$: two run together, side by side

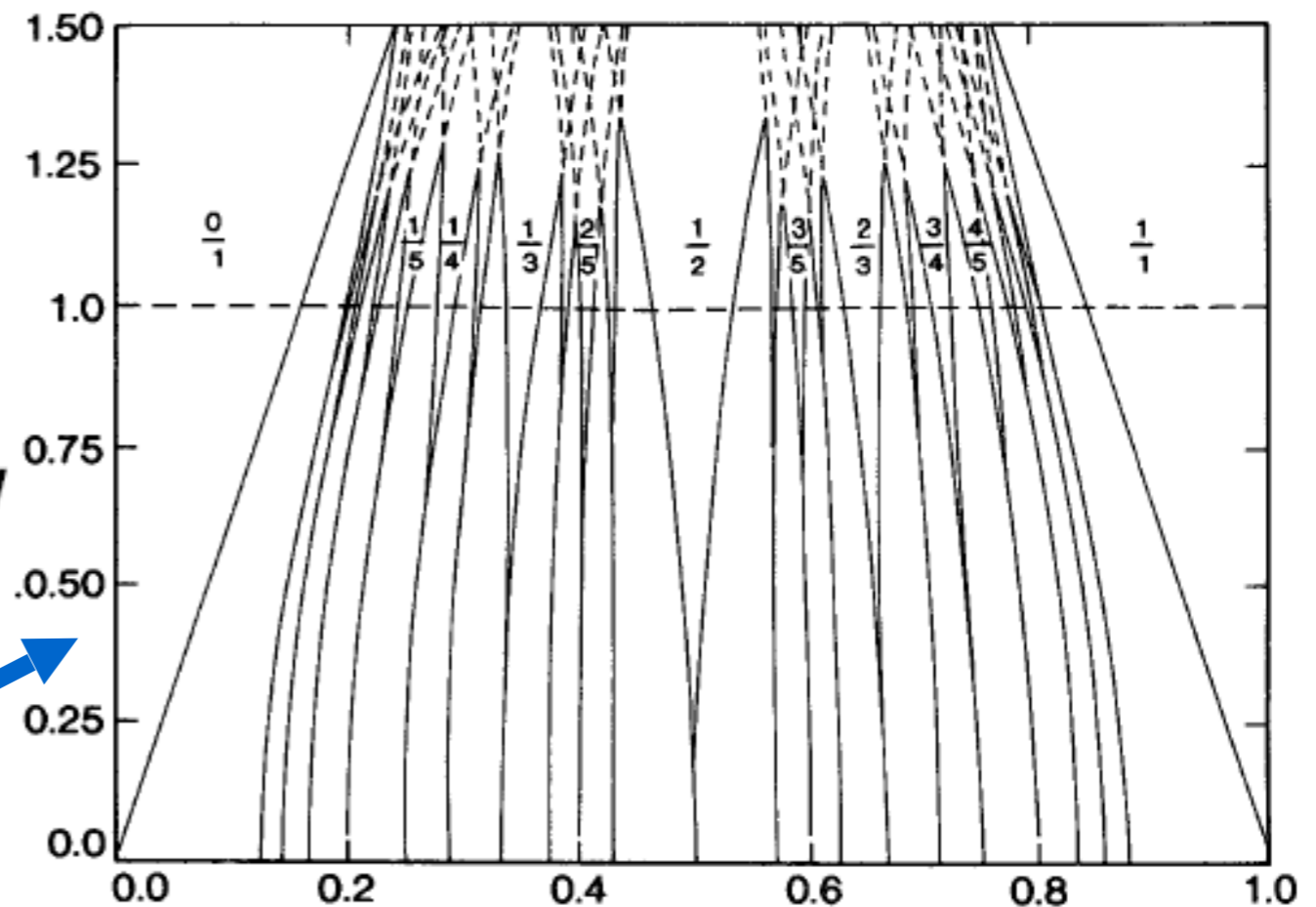
$0 < |\omega_1 - \omega_2| < K_1 + K_2$: run @ same speed, but separated by a constant distance (Strogatz 94)

Two “coupled” runners in a circular stadium

Simplest model of a periodically forced pendulum:



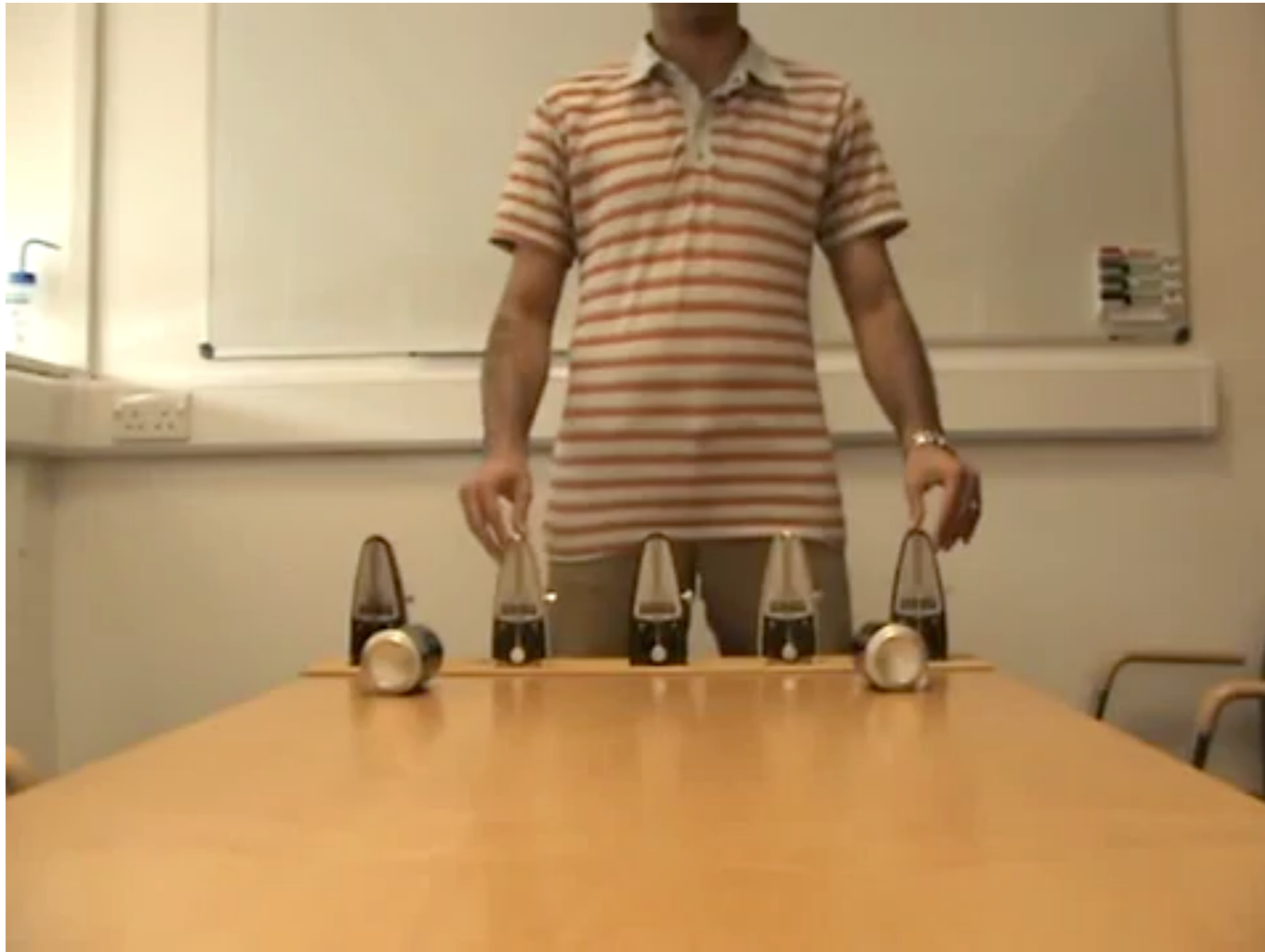
$$\omega / \omega_f = p / q$$



Circle map & Arnold Tongues

$$\theta_{n+1} = f_{\Omega}(\theta_n) = \theta_n + \Omega - \frac{K}{2\pi} \sin(2\pi\theta_n)$$

Another demonstration of nonlinear phase locking



<https://www.youtube.com/watch?v=W1TMZASCR-I>

Alireza Bahraminasab

Another demonstration of nonlinear phase locking



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Alireza Bahraminasab

Phase locking of glacial cycles

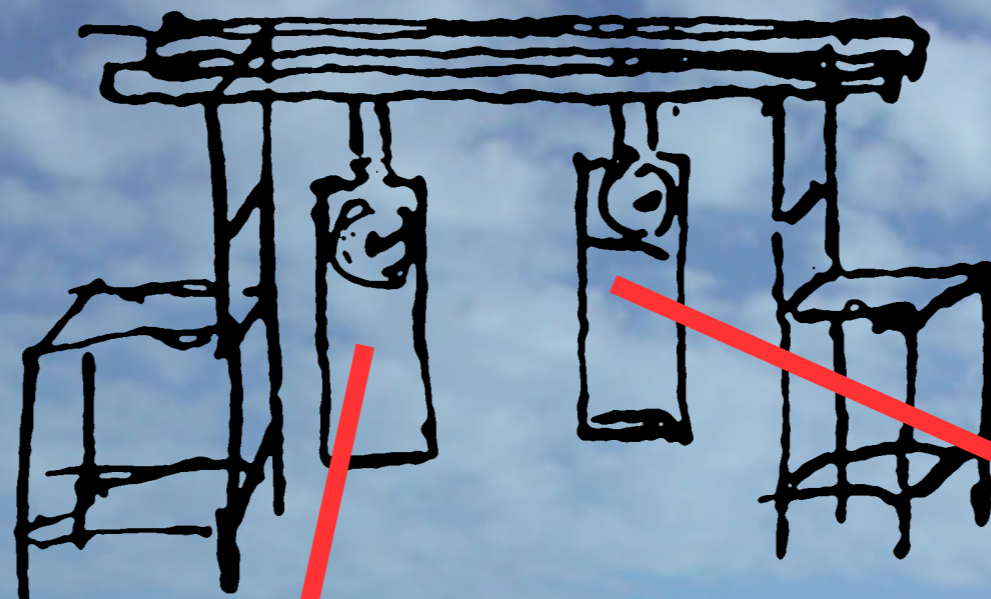
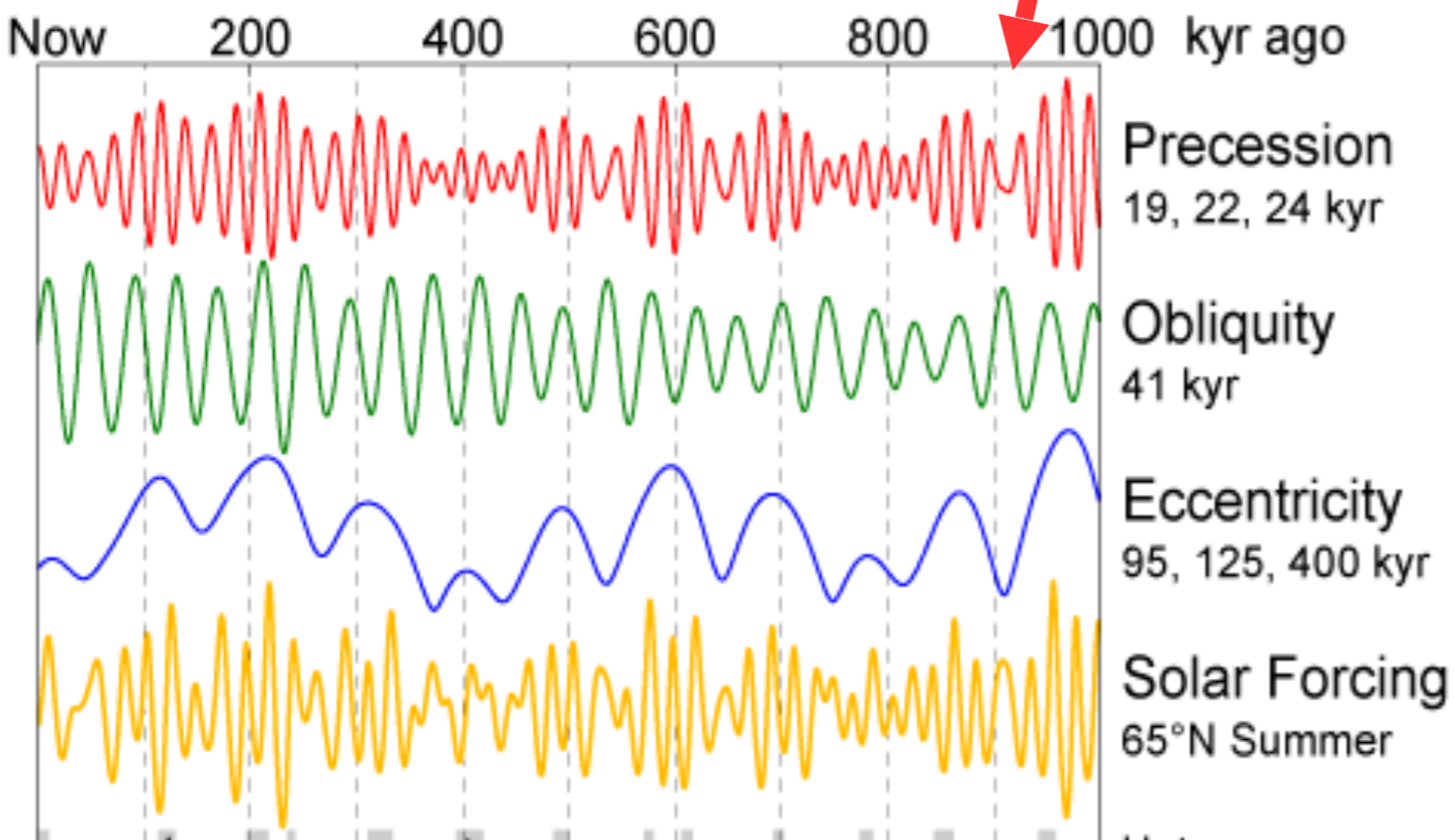


Diagram by Christian Huygens (1665)



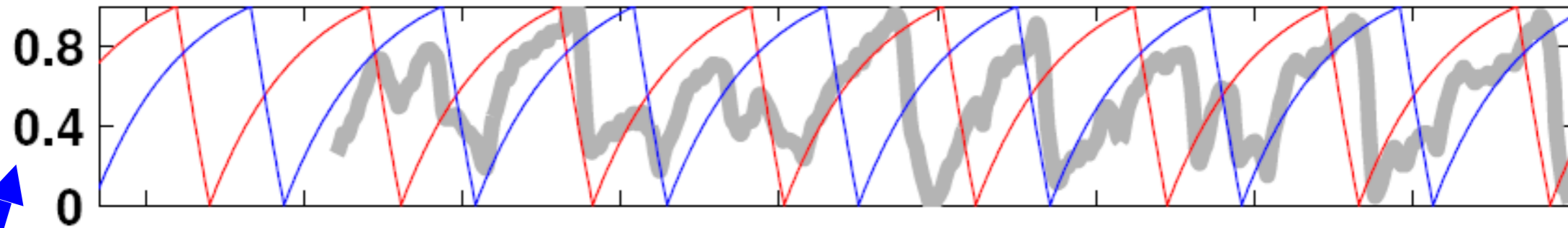
Why is it so simple to fit the ice volume proxy record?

Because glacial cycles are phase locked to Milankovitch forcing, and so are the models; **not** because the model's mechanism is correct...

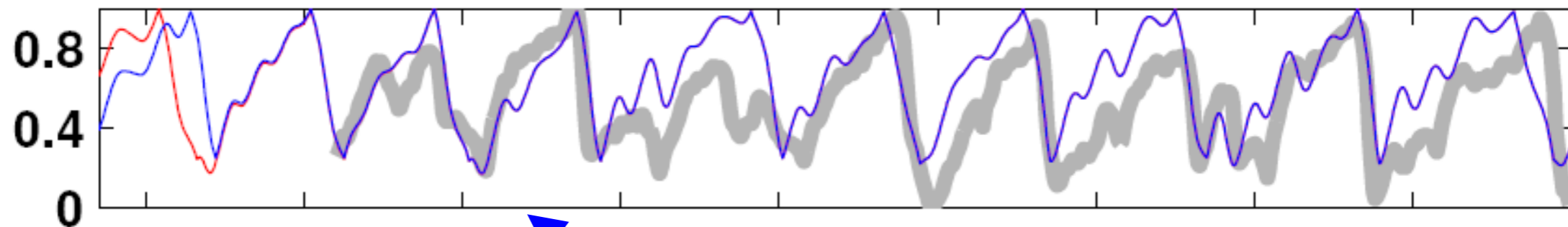
➔ **Try to better understand nonlinear phase locking...**

Phase locking: the mechanism

(a) no Milankovitch, no noise



(b) 65N, no noise



(a) model solution(s) w/o Milankovitch forcing

(b) with Milankovitch forcing, all initial conditions converge to a single time series that fits the observed ice volume record

Required ingredients:

Nonlinear oscillator(s): that can change its frequency as function of its amplitude);

Dissipation: “erases” memory of initial conditions & enables phase locking (radiative cooling in atm, glacial basal friction, ocn viscosity...)

Phase between Milankovitch and ice volume

“Recent well dated proxies show that the phase of Milankovitch forcing during the terminations of 18 kyr BP and 135 kyr BP are not the same...” [Gallup et al, Science 2002]



Perhaps Milankovitch forcing doesn't **have** to be the same during all terminations if Milankovitch forcing **paces** rather than **forces** the cycles?

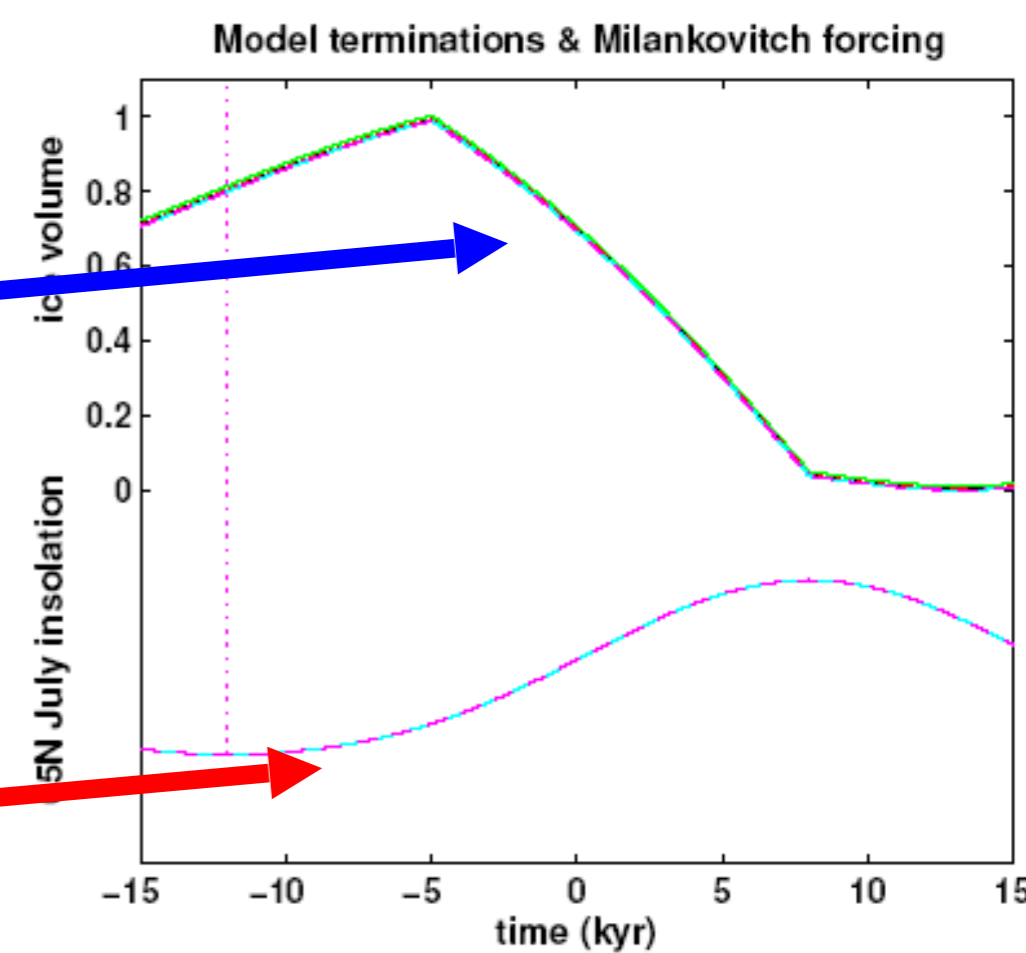
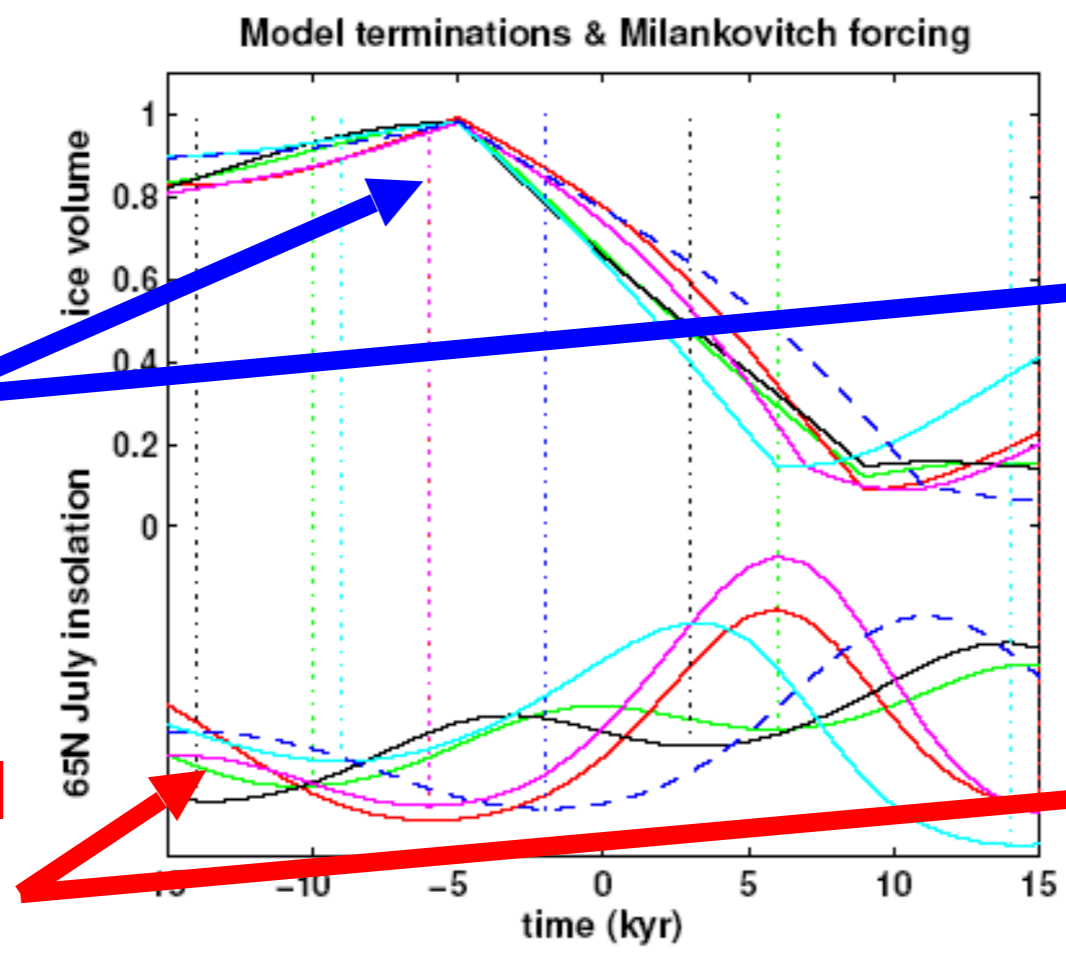
Phase between Milankovitch forcing and glacial terminations

phases during a model run forced by 65N July radiation:

And by a pure 40 kyr sine wave insolation curve

Model ice volume during all terminations

Milankovitch forcing during all terminations

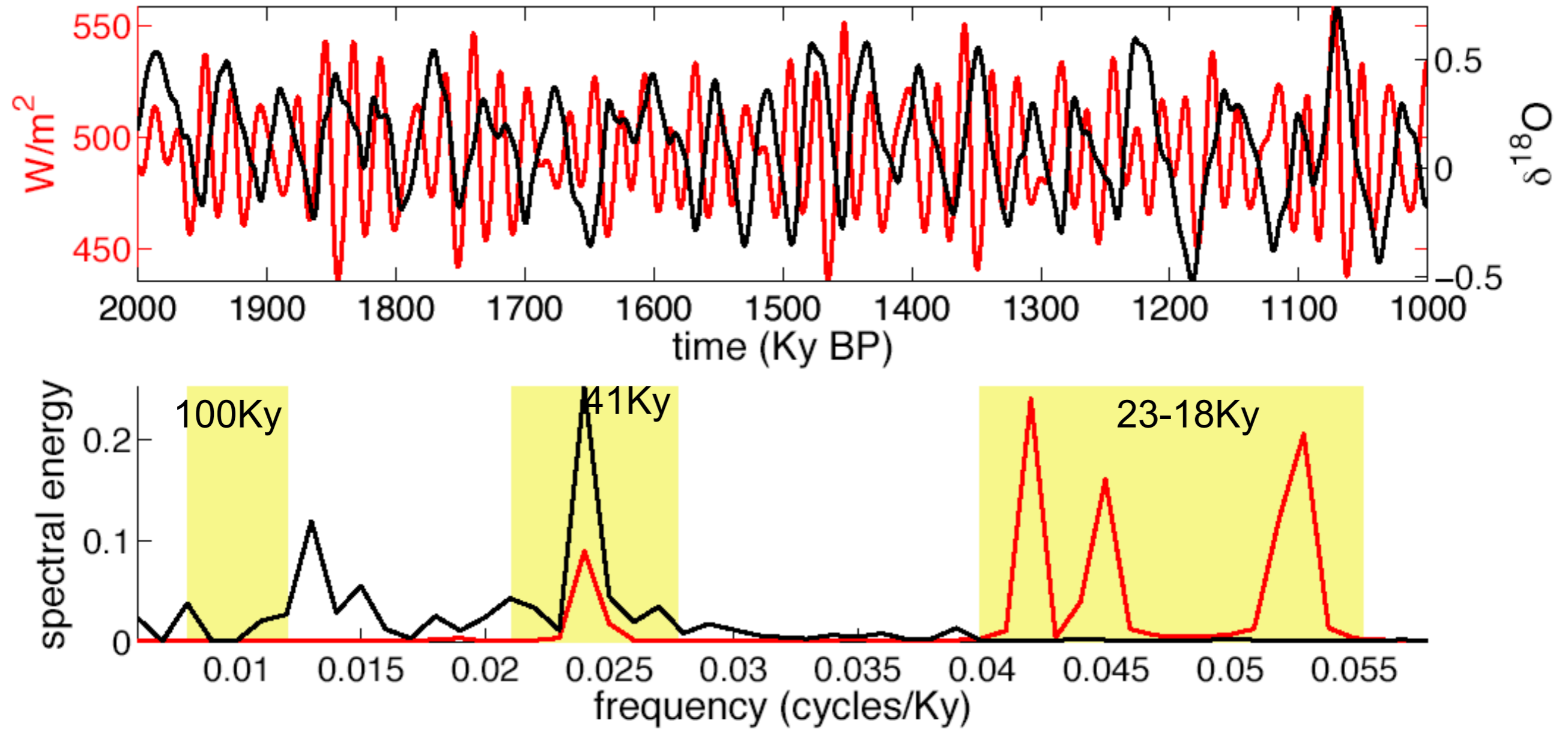


➡ Phase of Milankovitch forcing is not the same during different terminations, due to the irregular structure of the Milankovitch forcing. Still, Milankovitch forcing sets time of terminations via phase locking!

The 41 Ky problem and integrated insolation

The 41Ky problem

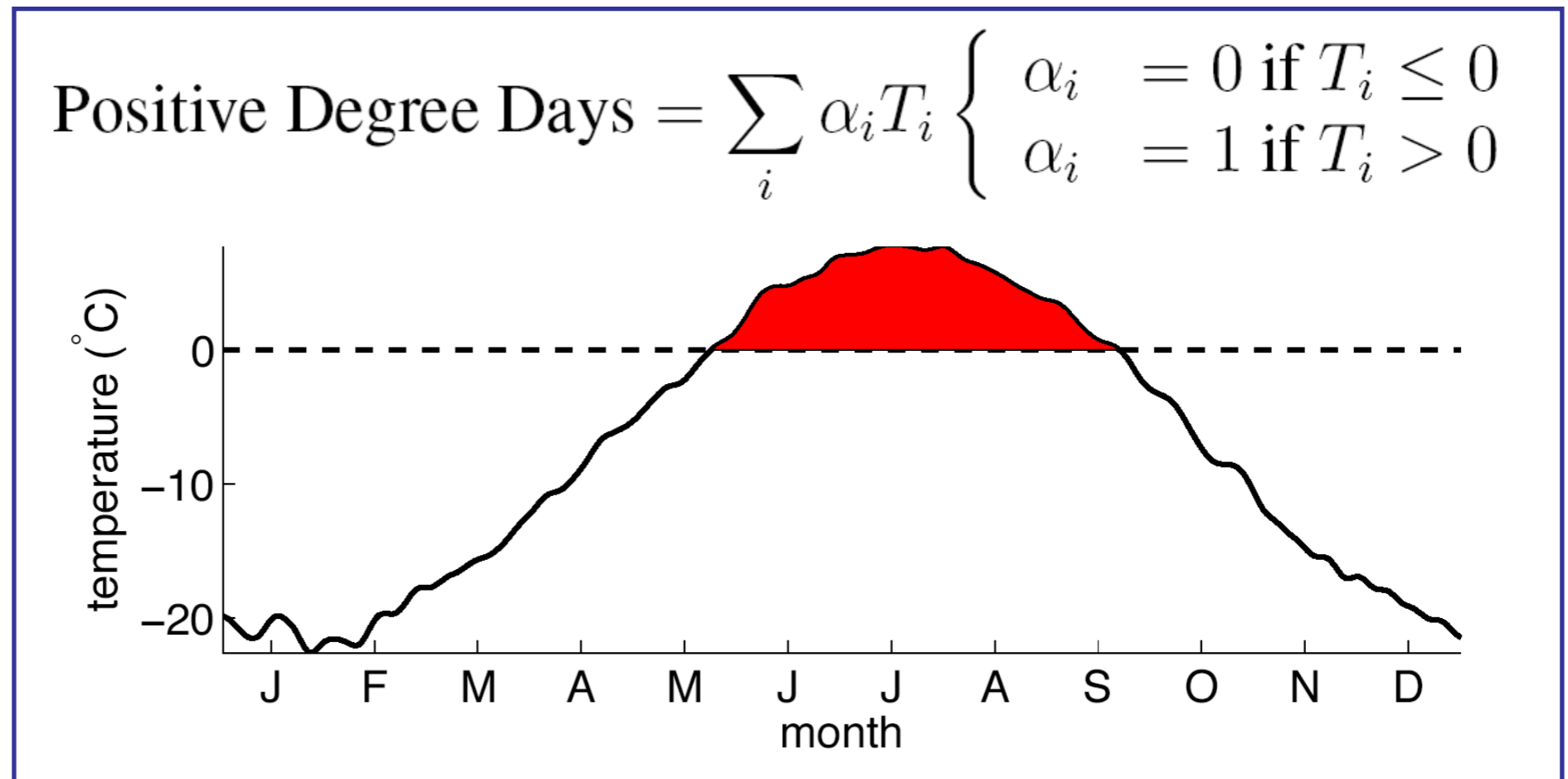
Imbrie et al (1992), Muller and MacDonald (2001), Nisancioglu (2004), and many others



Representing the relationship between insolation, temperature and ablation

Braithwaite (1984), Paterson (1994)

ablation is known to be correlated with PDD



assuming insolation determines surface ice temperature, define summer energy similarly to PDD,

$$\text{Summer Energy} = \sum_i \beta_i W_i \begin{cases} \beta_i = 0 & \text{if } W_i \leq \tau \\ \beta_i = 1 & \text{if } W_i > \tau \end{cases}$$

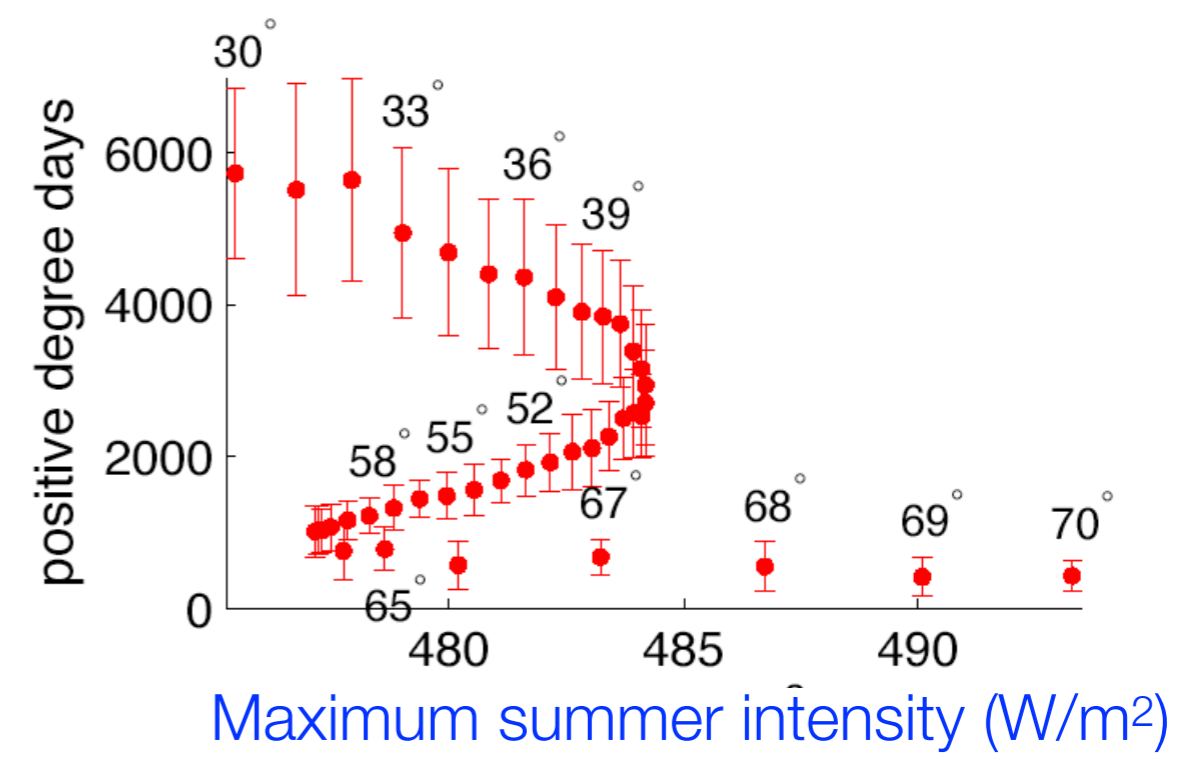
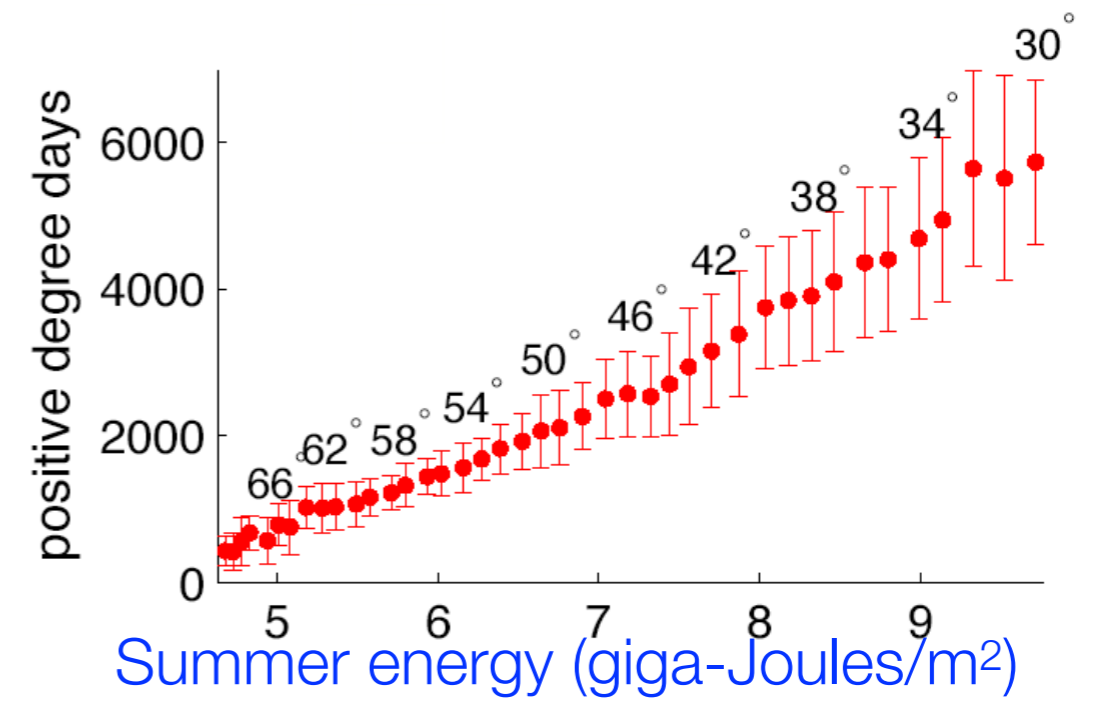
In class workshop

Do we expect the insolation threshold τ that makes summer energy related to melting to be larger or smaller for a warmer climate?

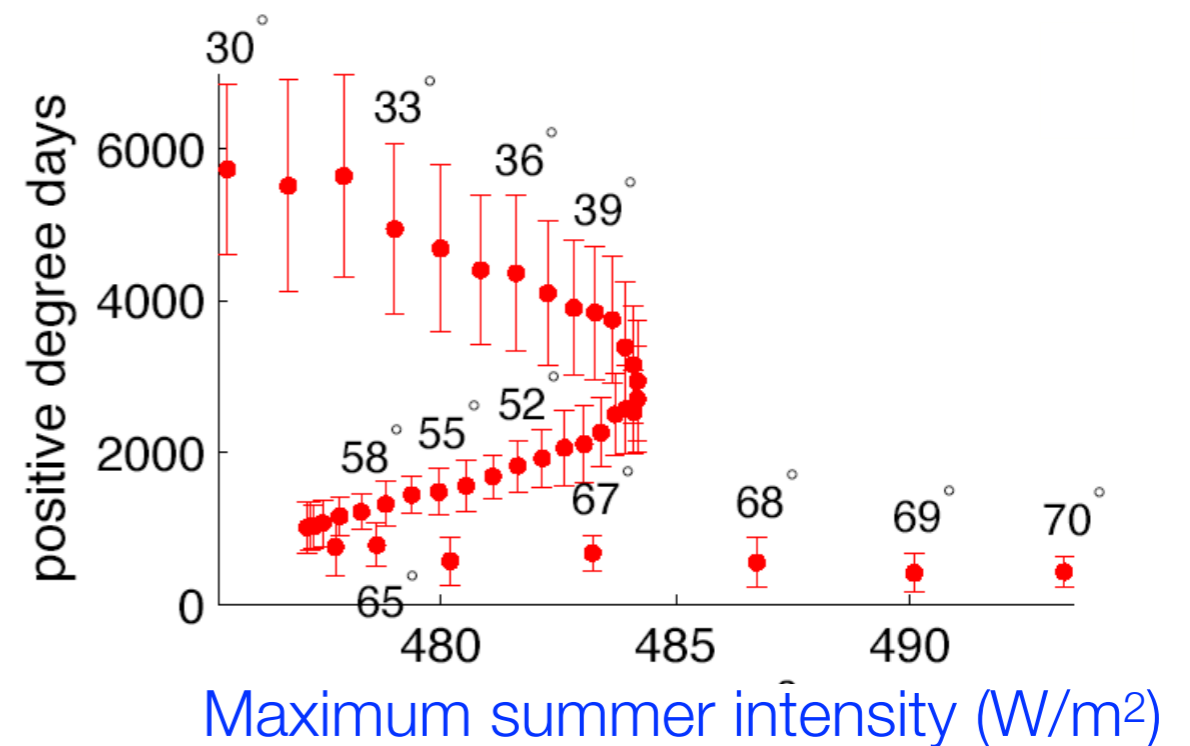
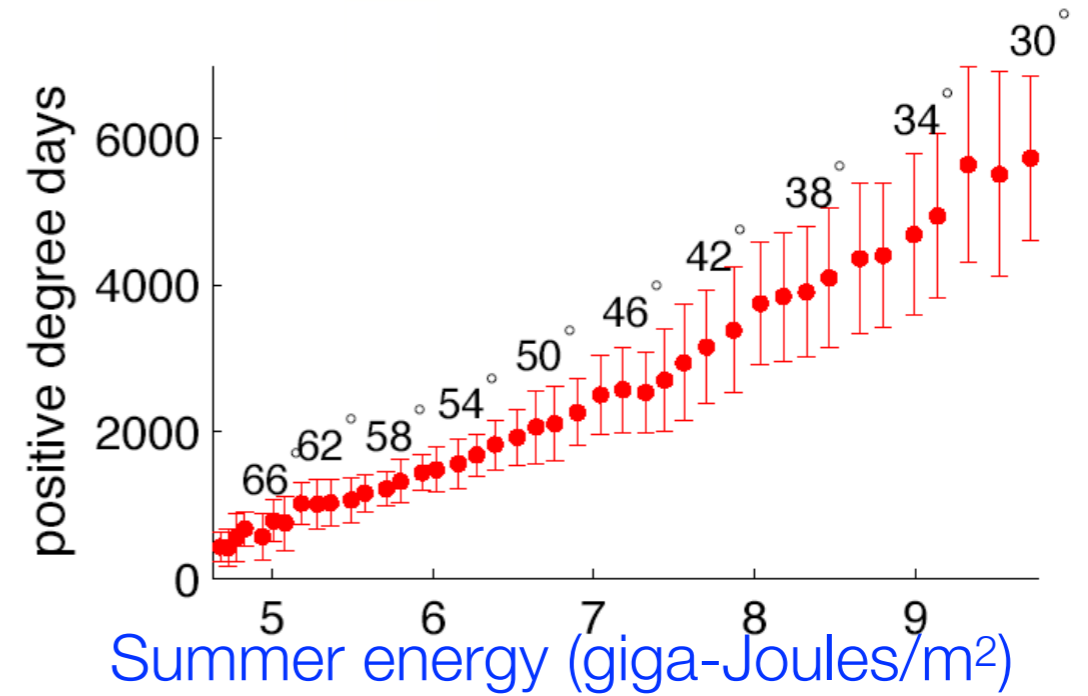
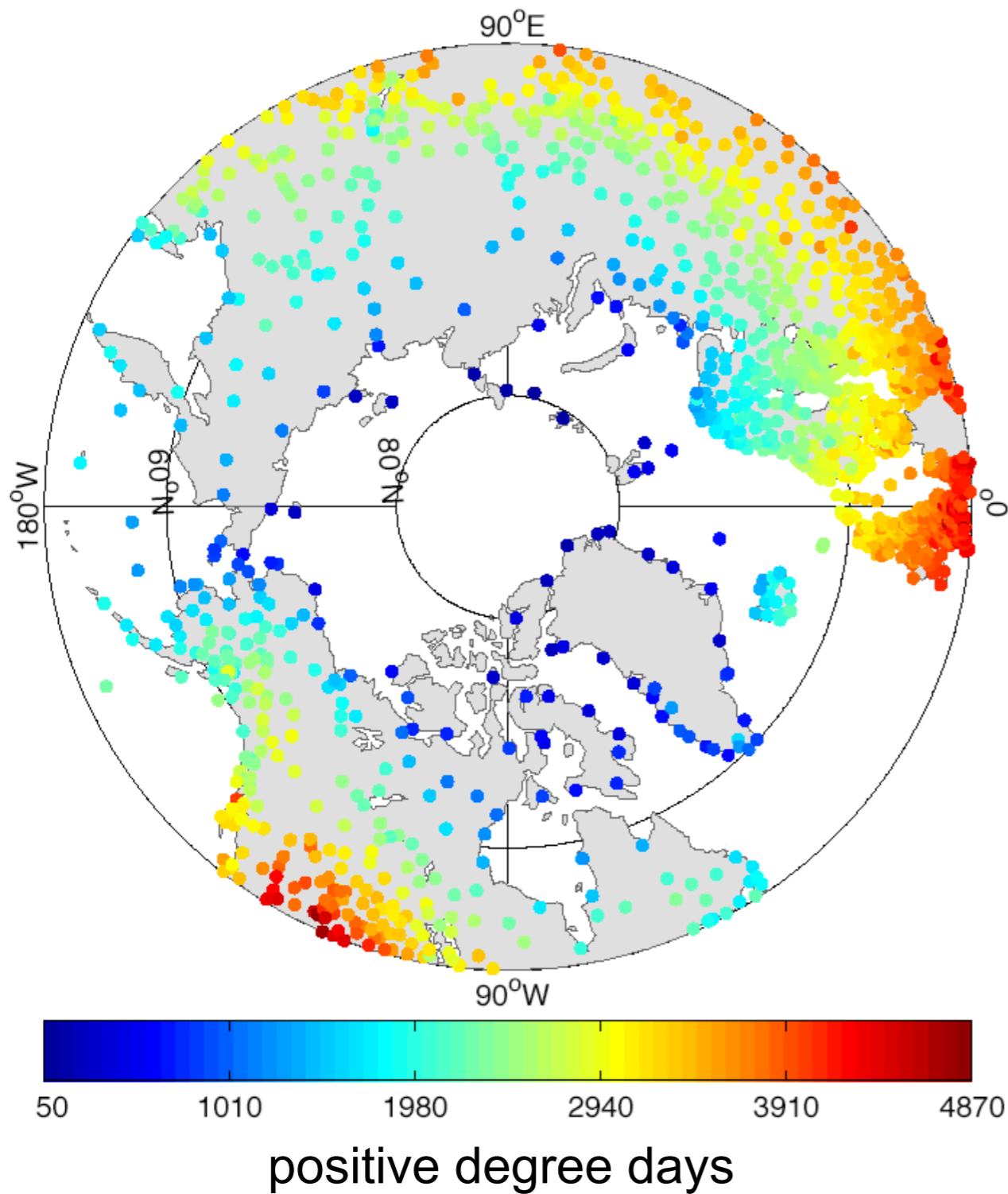
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[answer in presenter notes]

Summer energy is strongly correlated with Positive degree days and insolation, but maximum summer insolation is not!



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Milankovitch forcing: precession+eccentricity

In-class workshop

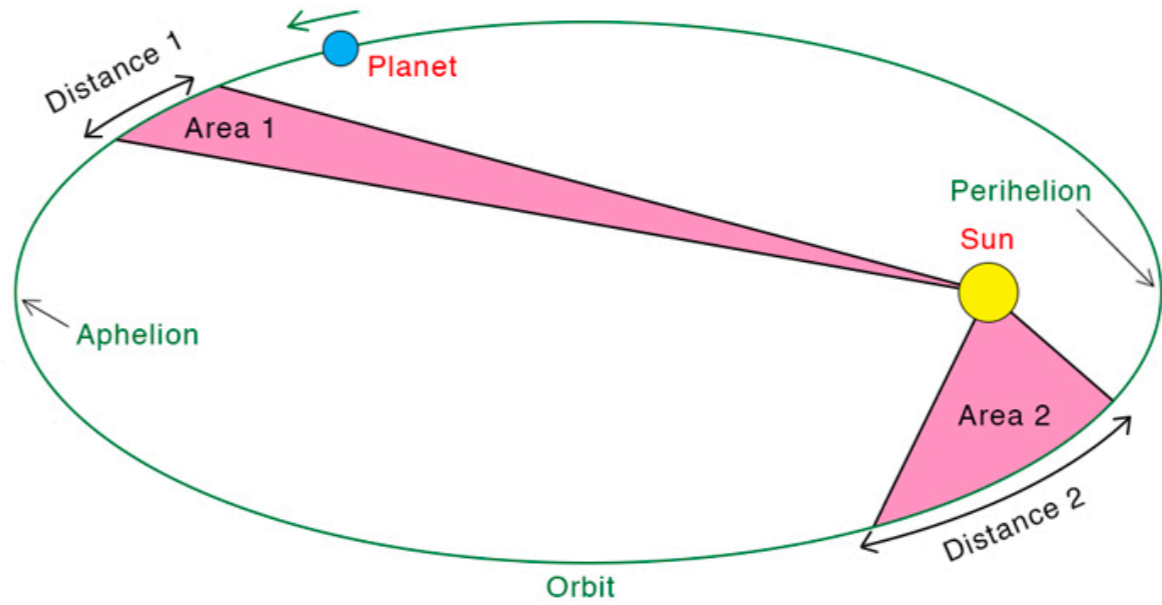
Use Kepler's laws to explain *qualitatively* why the integrated summer insolation above a threshold (summer energy) is expected to be independent of precession.

Kepler Laws:

- (1) Planets move in elliptical orbits with the Sun as a focus,
- (2) A planet covers the same area of space in the same amount of time no matter where it is in its orbit

Kepler's Second Law

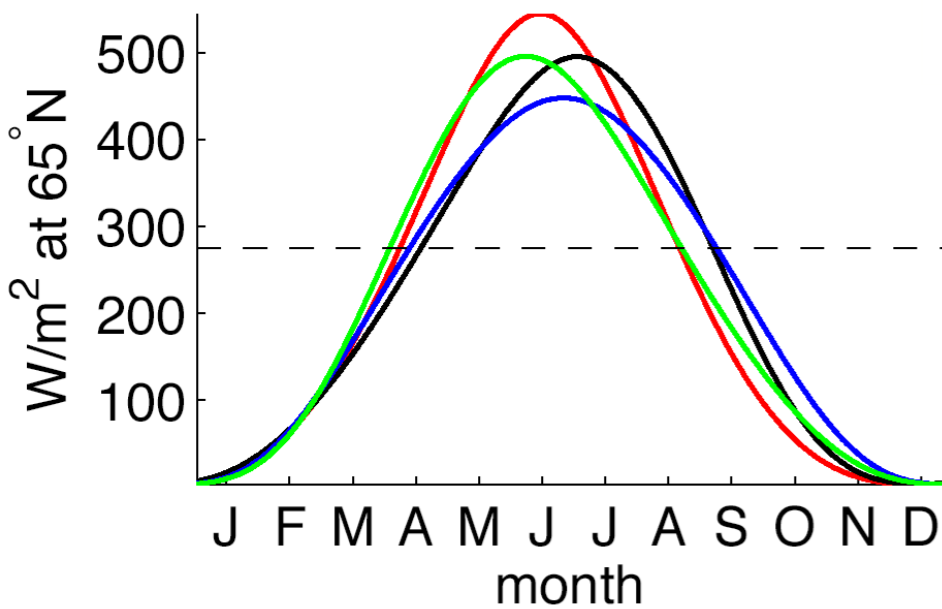
A planet sweeps out equal areas in equal intervals of time



$$\text{Time taken to travel distance 1} = \text{Time taken to travel distance 2} \Rightarrow \text{Area 1} = \text{Area 2}$$

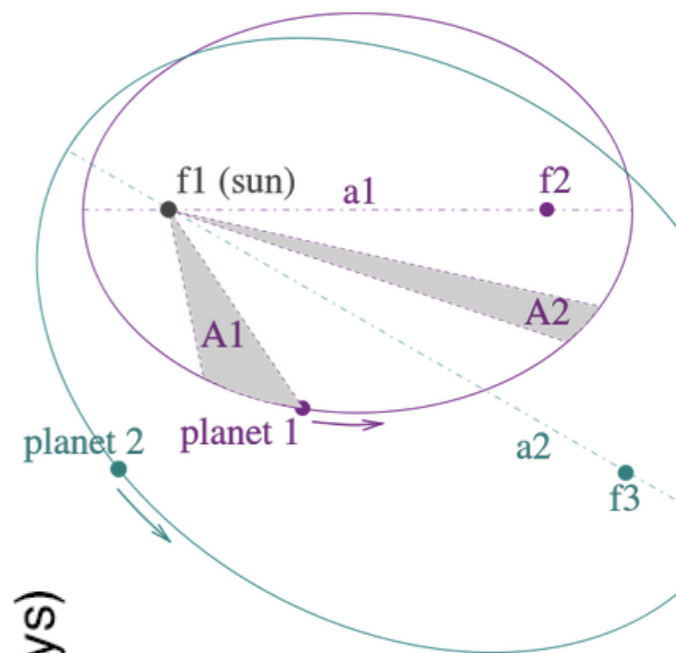
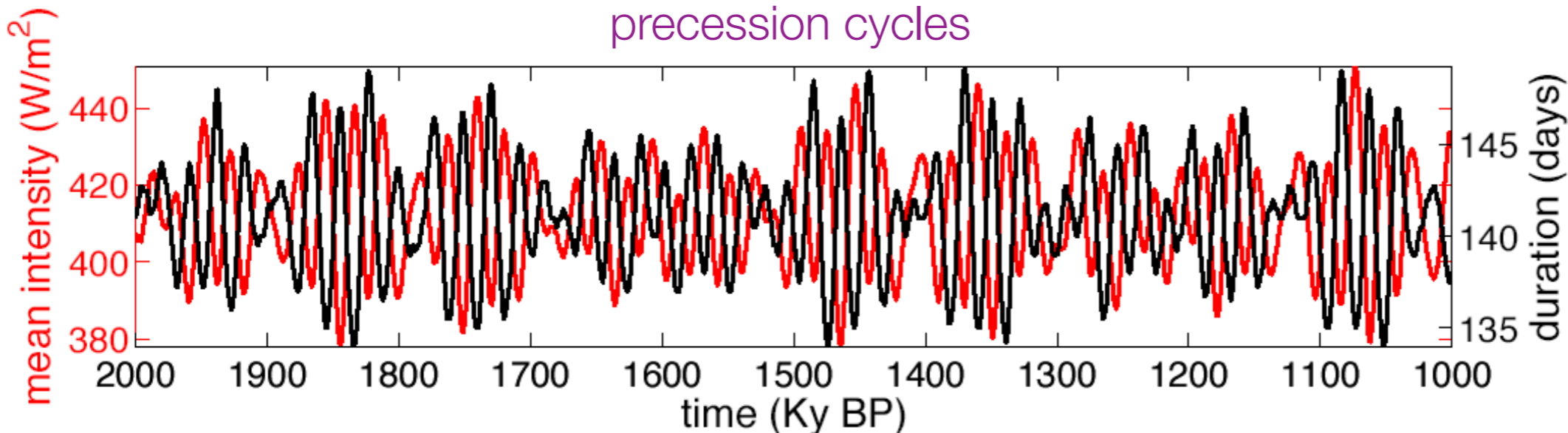
[schematic from <https://www.sciencefacts.net/keplers-laws.html>]

Summer intensity and duration

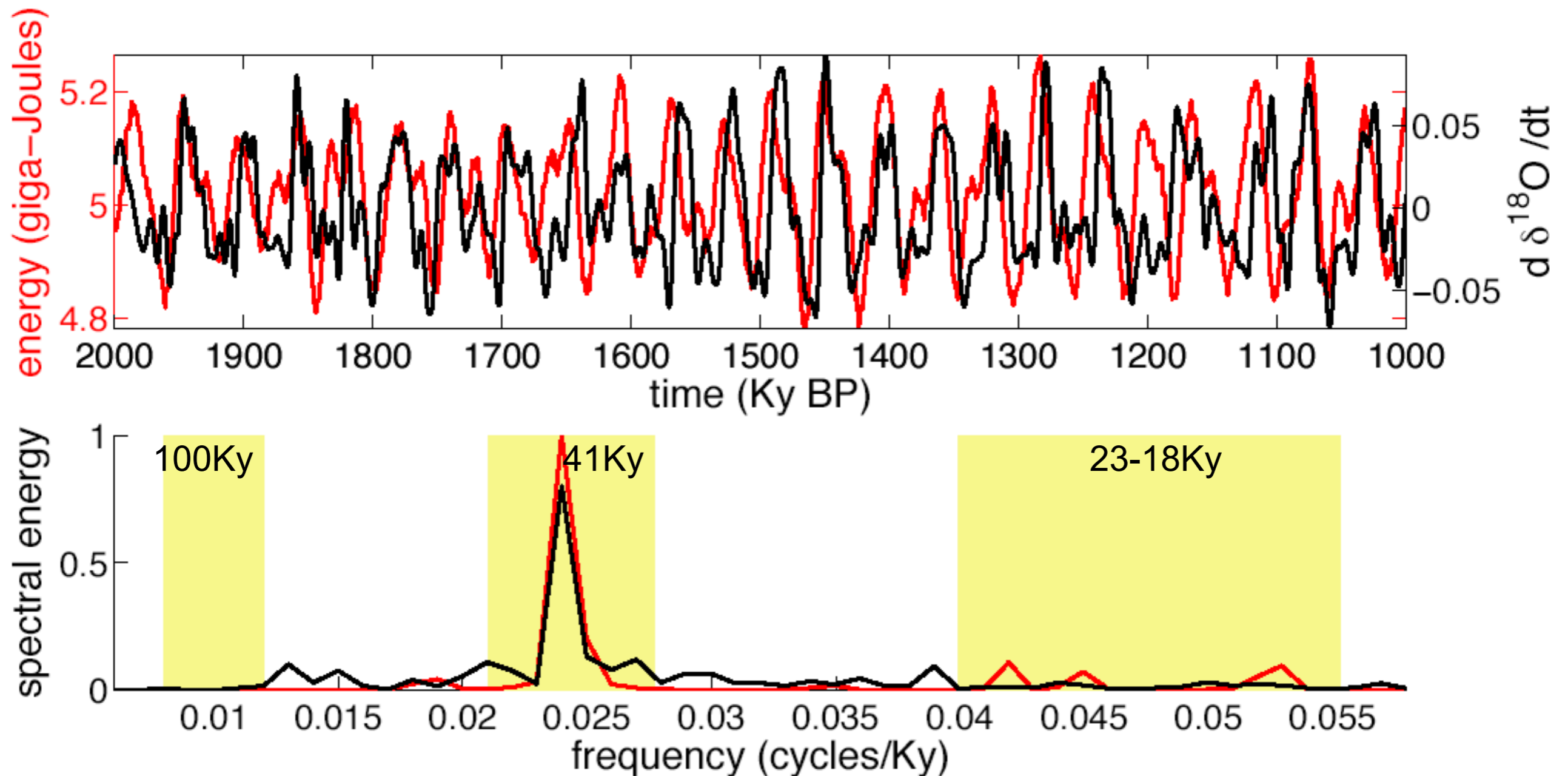


Summer duration & intensity are anti-correlated: just when Earth is closest to the sun during summer, summertime is the shortest (Kepler*). When intensity is integrated over the summertime, precession-related changes in duration and intensity nearly balance one another, and the obliquity component is dominant. (*) a line connecting a planet & the sun sweeps out equal areas during equal intervals of time, due to angular momentum conservation

Insolation intensity & summer duration are anti-correlated during precession cycles



Early Pleistocene summer energy and the rate of ice-volume change



High threshold: integrated insolation=peak summer intensity, precession dominates; **Low threshold:** obliquity dominates.

Cold climate: need a high threshold for representing melting temperature;

Warm climate: need only a low threshold.

→ Relevant to the mid-Pleistocene transition from 41kyr to 100kyr ??

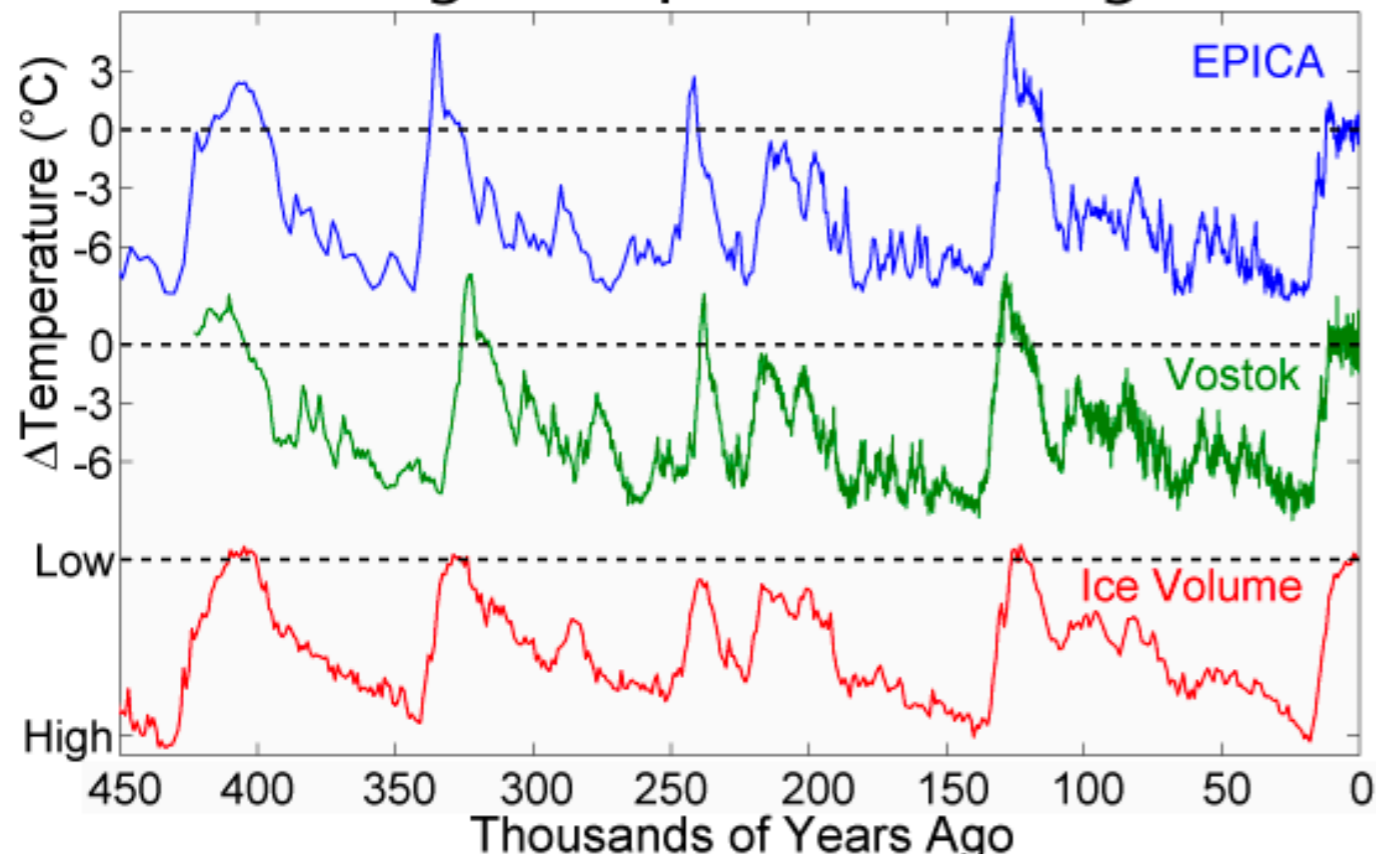
Milankovitch forcing: what might have happened during the mid-Pleistocene transition that led to the transition from 41 to 100 kyr?

In-class workshop

Why did summer energy stop dictating the ice volume evolution?

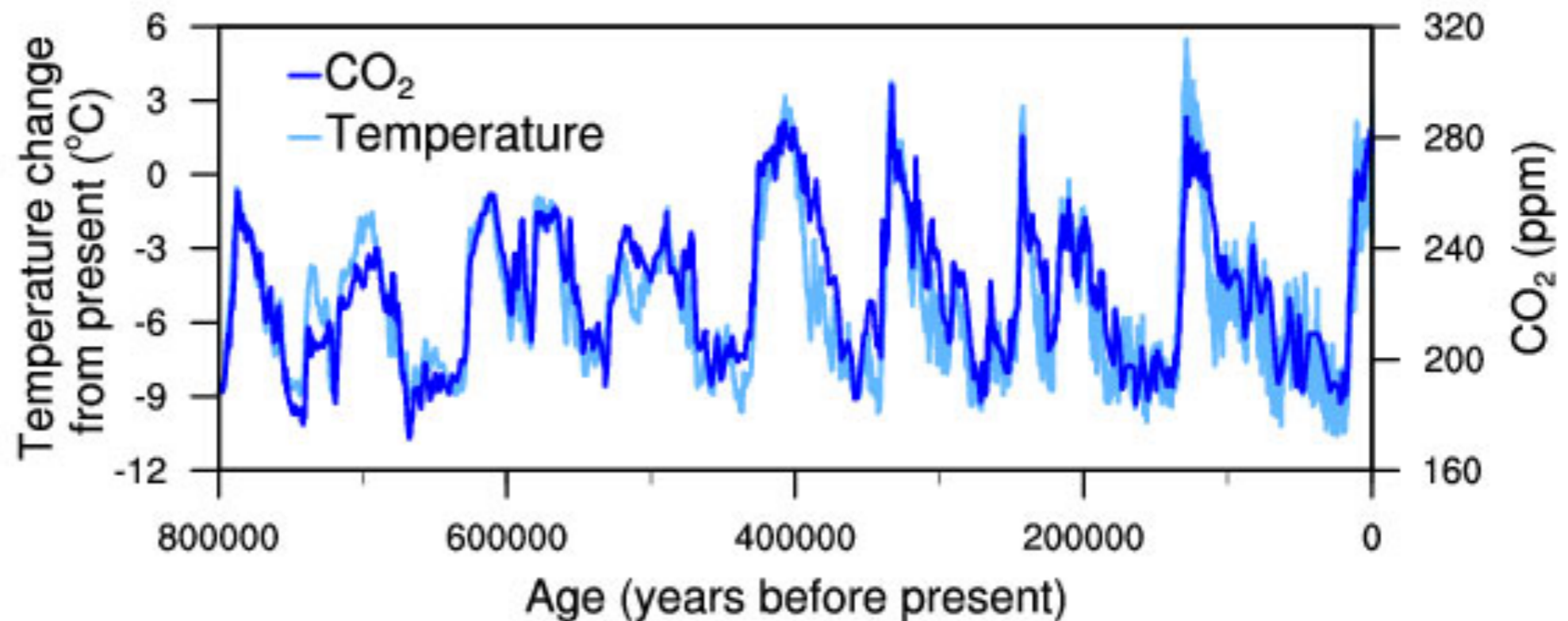
Glacial CO₂ variations

Ice Age Temperature Changes



Variations of temperature, ice volume and CO₂

https://commons.wikimedia.org/wiki/File:Ice_Age_Temperature.png



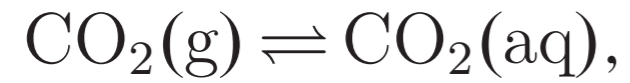
<https://commons.wikimedia.org/wiki/File:Temperature-change-and-carbon-dioxide-change-measured-from-the-EPICA-Dome-C-ice-core-in-Antarctica-v2.jpg>

notes

The ocean carbonate system

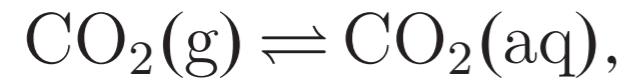
The ocean carbonate system

Atmospheric $\text{CO}_2(\text{g})$ is in equilibrium with dissolved $\text{CO}_2(\text{aq})$, (Henry's Law)



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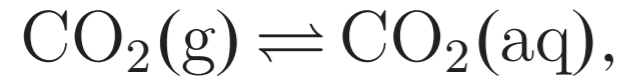


its reaction with water is given by



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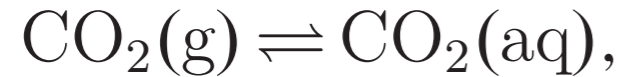


treated together as a single variable



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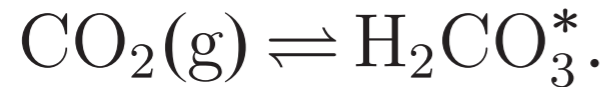
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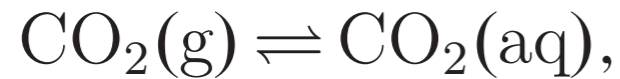


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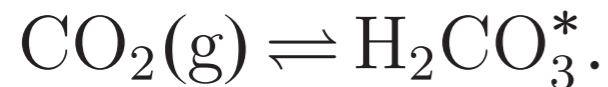
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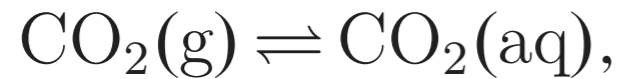


Carbonic acid releases two protons



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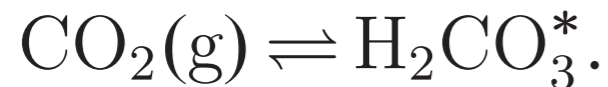
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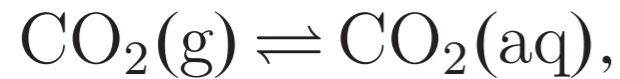


Water dissociation



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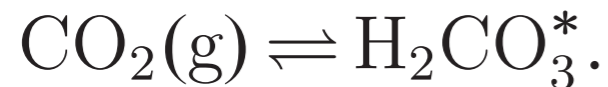
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Water dissociation



$$K_H \equiv K_0(T, S, p) = \frac{[\text{H}_2\text{CO}_3^*]}{[\text{CO}_2(\text{g})]}$$

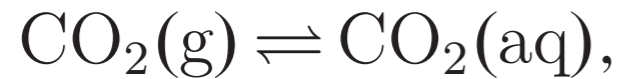
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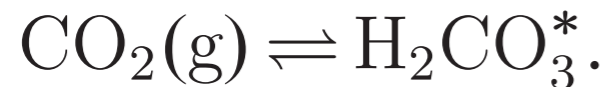
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➔ **6 unknown** $[\text{CO}_2(\text{g})]$, $[\text{H}_2\text{CO}_3^*]$, $[\text{OH}^-]$, $[\text{H}^+]$, $[\text{HCO}_3^-]$, $[\text{CO}_3^{2-}]$, **and only four equations**

Complete equations for the ocean carbonate system

Charge conservation (alkalinity)

$$\text{Alk} = [\text{HCO}_3^-] + 2[\text{CO}_3^{2-}] + [\text{OH}^-] - [\text{H}^+]$$

← Carbonate alkalinity Alk_C →

T, S, p : the ocean temperature, salinity, and pressure where the carbonate system is solved.

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The next slide shows these to be small at the oceanic pH range

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Carbonate system solution

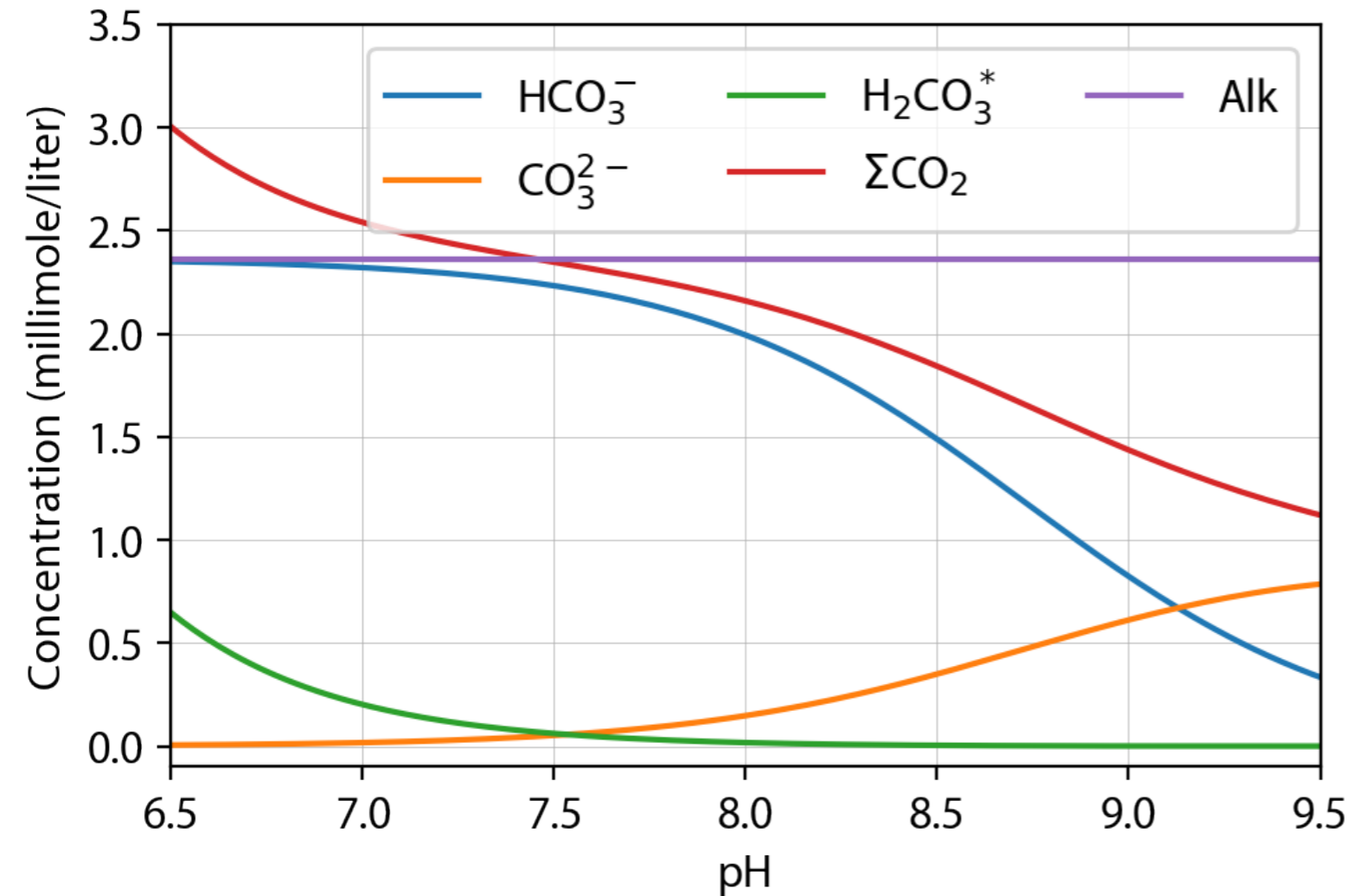
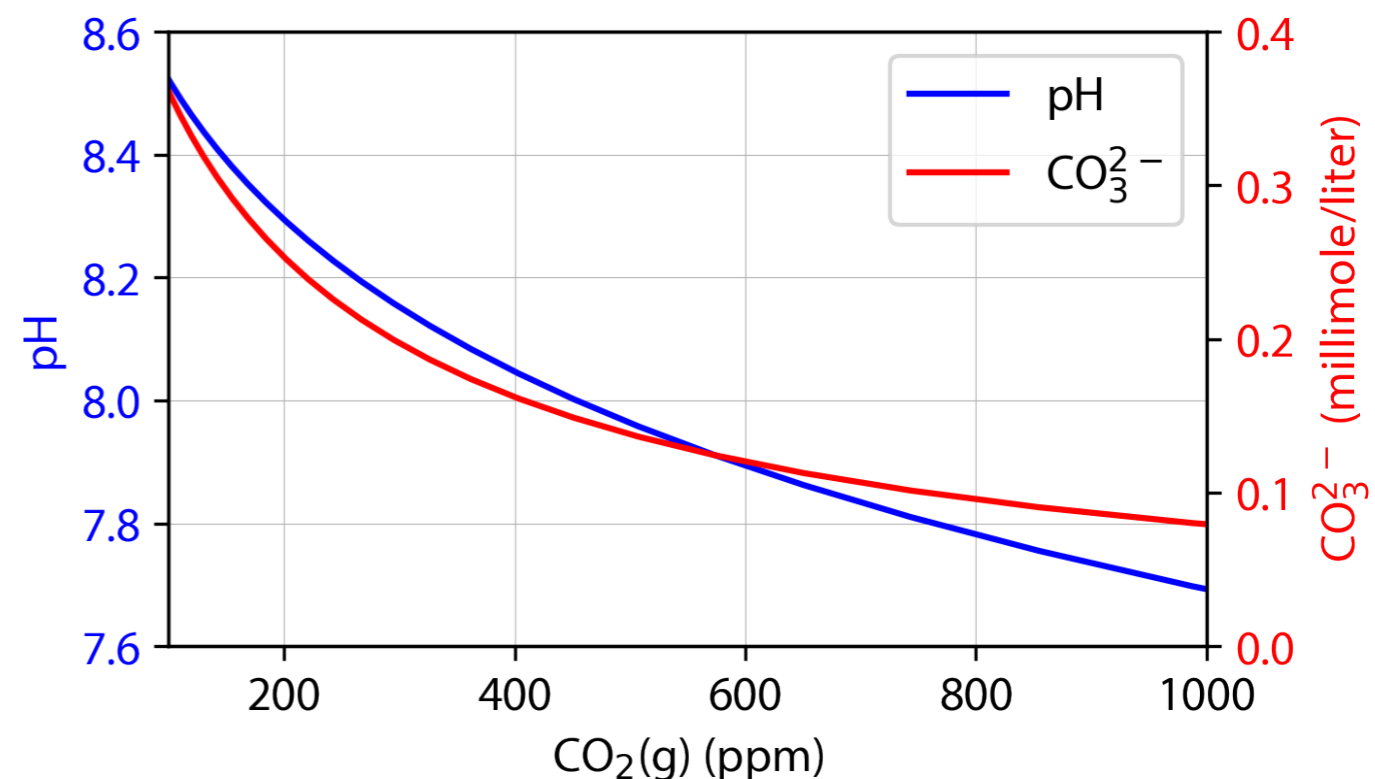


Figure 5.3: The solution of the carbonate system, showing the concentration of carbonate species as a function of pH for a fixed alkalinity.

Figure 5.4: The response of pH and carbonate ion to CO_2 increase.

The solution of the carbonate system for a fixed alkalinity as in Figure 5.3, showing the ocean pH (blue) and the carbonate ion CO_3^{2-} concentration (red) as a function of atmospheric CO_2 .



Carbonate system solution

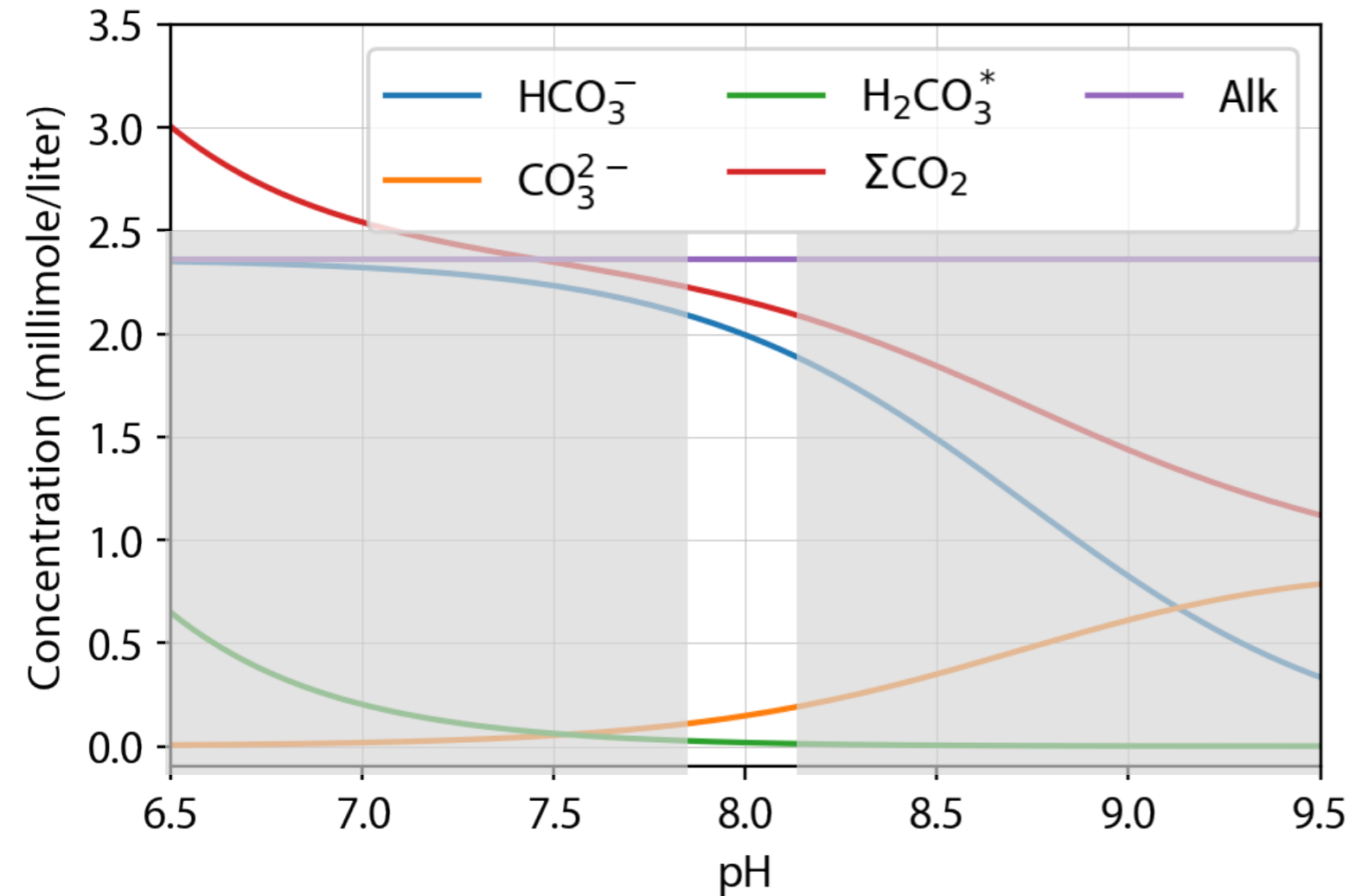
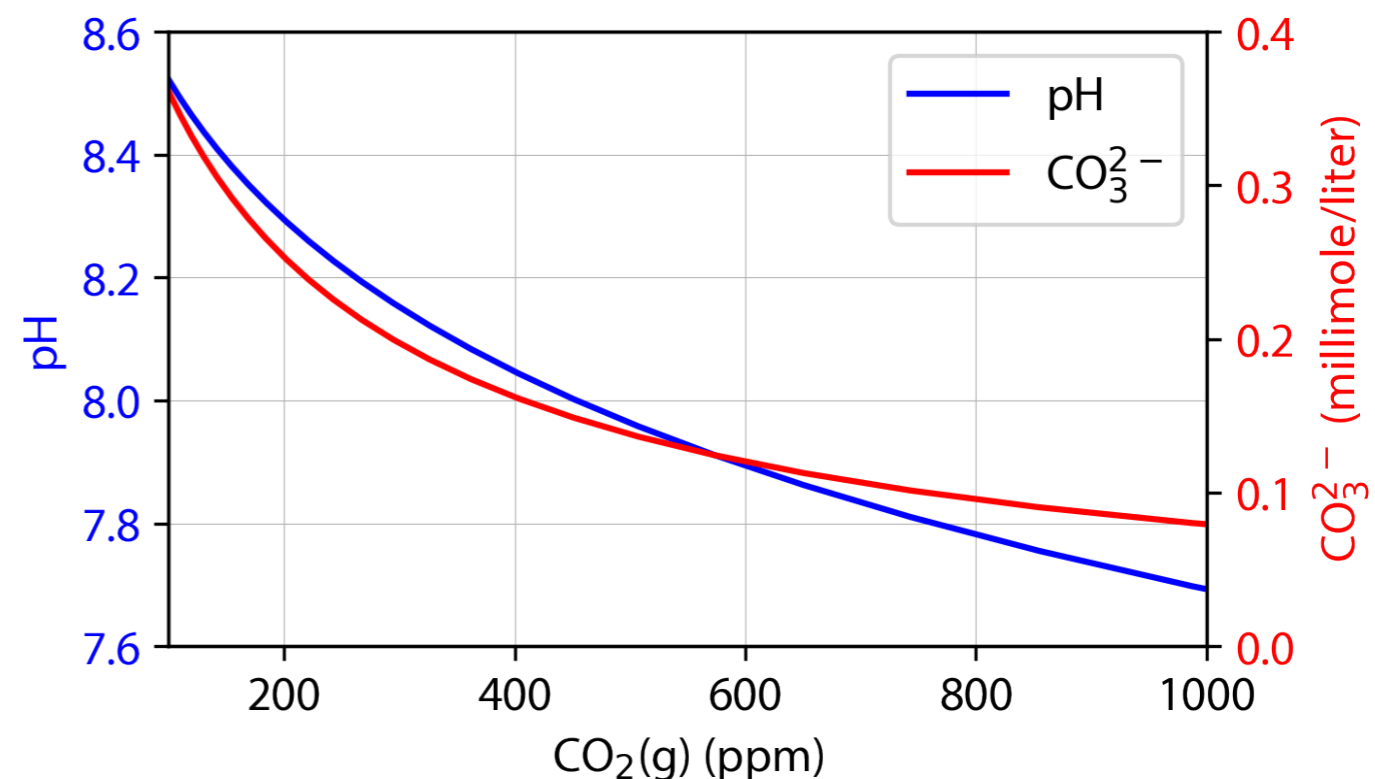


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In-class workshop

Solve for HCO_3^- , CO_3^{2-} and H^+ using the approximate equations.

Approximate solution of the carbonate system

For the values of pH at present/near future:



Approximate solution of the carbonate system

For the values of pH at present/near future:



So that

$$\text{Alk}_C = [\text{HCO}_3^-] + 2[\text{CO}_3^{2-}] + \cancel{[\text{OH}^-]} - \cancel{[\text{H}^+]}$$

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Approximate solution of the carbonate system

For the values of pH at present/near future:

$$[\text{HCO}_3^-], [\text{CO}_3^{2-}] \gg [\text{H}^+], [\text{OH}^-], [\text{H}_2\text{CO}_3^*].$$

So that

$$\text{Alk}_C = [\text{HCO}_3^-] + 2[\text{CO}_3^{2-}] + \cancel{[\text{OH}^-]} - \cancel{[\text{H}^+]}$$

$$C_T = [\text{HCO}_3^-] + [\text{CO}_3^{2-}] + \cancel{[\text{H}_2\text{CO}_3^*]}.$$

➔ only 5 unknowns (OH^- drops out) & the eqns become

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Approximate solution of the carbonate system

For the values of pH at present/near future:

$$[\text{HCO}_3^-], [\text{CO}_3^{2-}] \gg [\text{H}^+], [\text{OH}^-], [\text{H}_2\text{CO}_3^*].$$

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$$\text{Alk}_C = [\text{HCO}_3^-] + 2[\text{CO}_3^{2-}] + \cancel{[\text{OH}^-]} - \cancel{[\text{H}^+]}$$

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what could be simpler 🤔

Response to warming

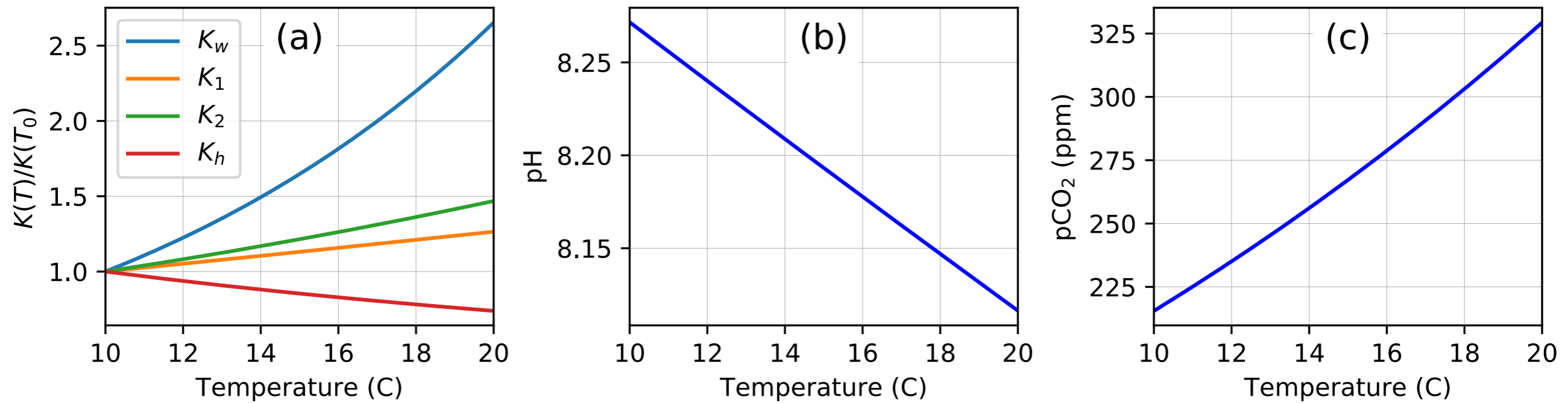


Figure 5.5: Response of the carbonate system to warming, as a function of the ocean temperature. The DIC and alkalinity are assumed fixed. (a) Reaction constants normalized by their values at 10 °C. (b) pH. (c) Atmospheric $p\text{CO}_2$.

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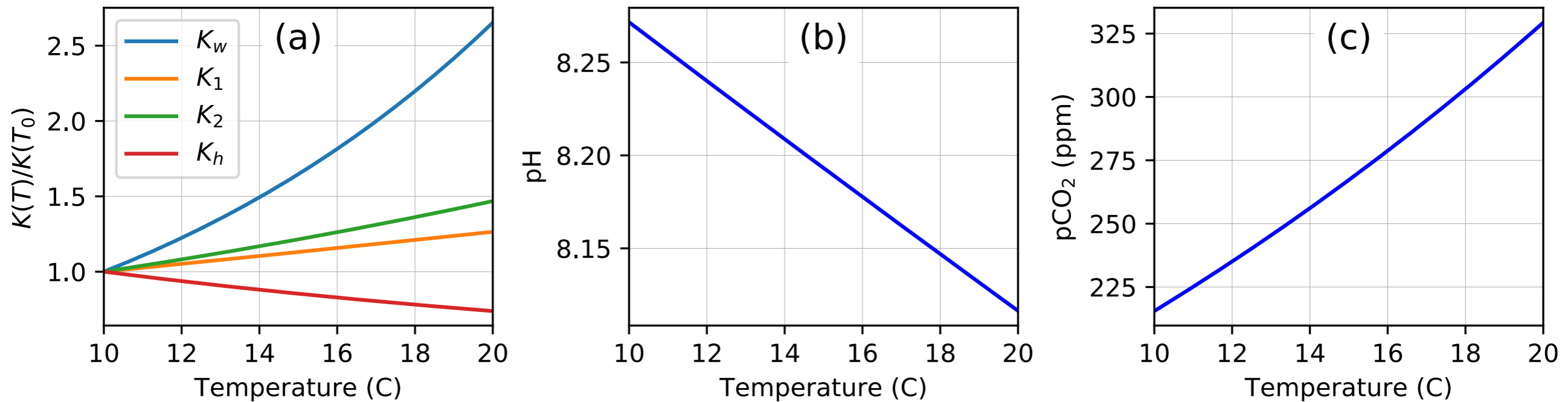


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$$[\text{H}^+] = K_2 \frac{2C_T - \text{Alk}_C}{\text{Alk}_C - C_T}, \quad [\text{CO}_2(\text{g})] = \frac{K_2}{K_1 K_H} \frac{(2C_T - \text{Alk}_C)^2}{\text{Alk}_C - C_T}.$$

K_1 , K_2 and K_H all play a role, not only Henry's constant responsible for the dissolution of CO_2 . In solution for the atmospheric CO_2 concentration, $K_2/(K_1 K_H)$, Henry's constant K_H decreases with temperature, while the other two increase. The ratio overall increases, leading to the increase in atmospheric CO_2 with warming.

Finally, attempting to understand the glacial CO₂ problem using a box model

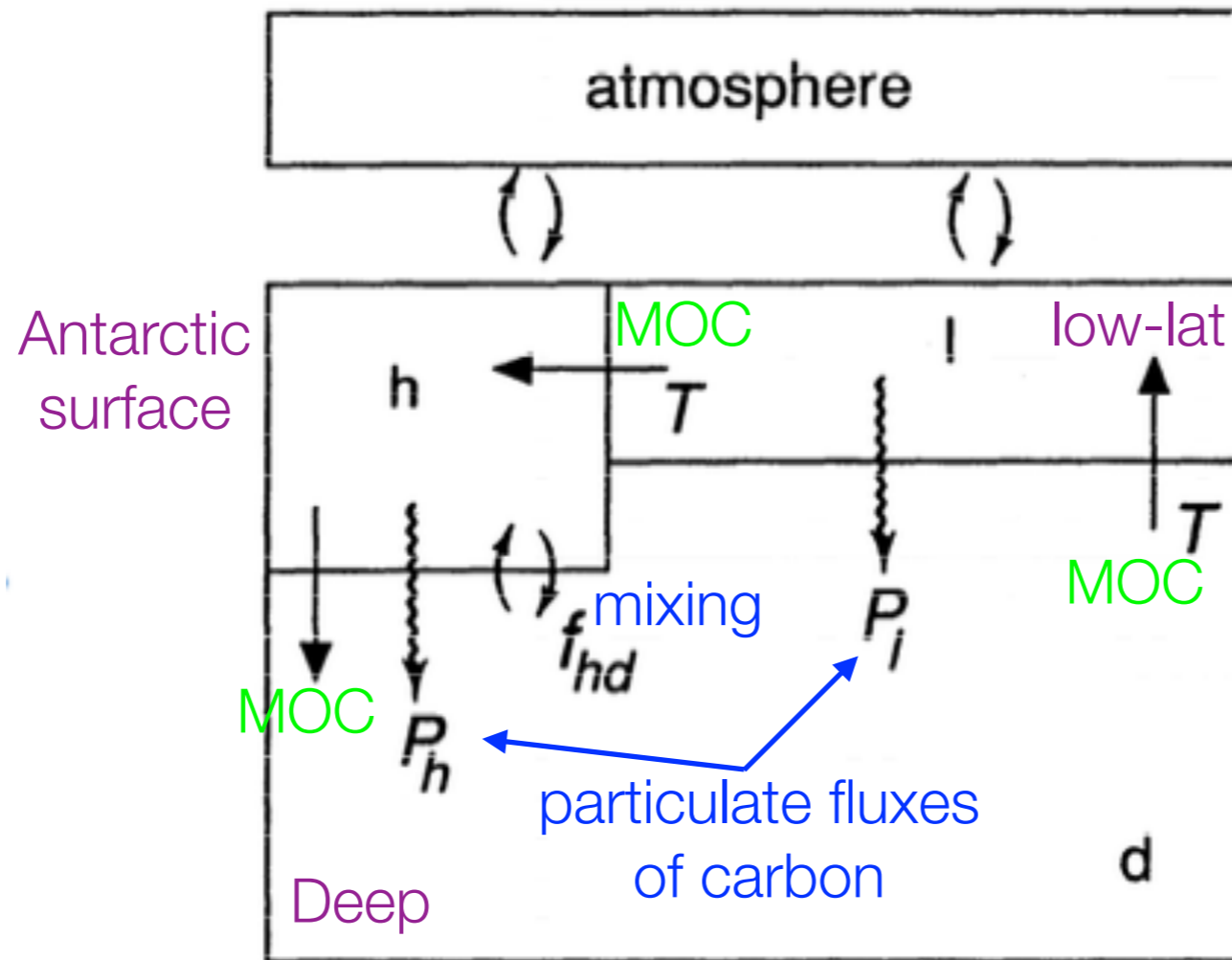


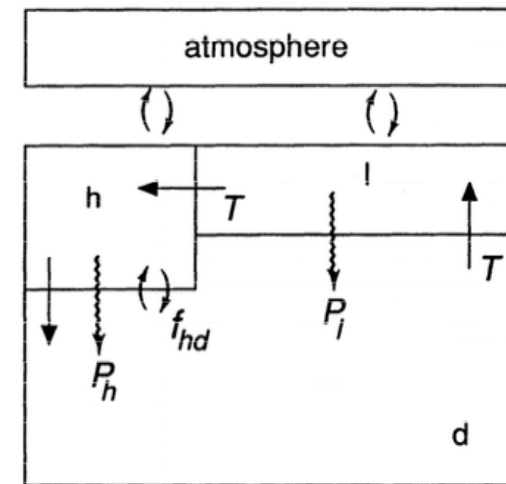
Figure 1. Schematic diagram of the three-box model of *Sarmiento and Toggweiler* [1984] and *Siegenthaler and Wenk* [1984].

Toggweiler 1999 model

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In the low-latitude surface box, assume that the upwelling nutrient flux is completely utilized by the biology. The downward carbon particulate flux (in moles of carbon) is then,

$$P_l = r_{c:p} \times T \times PO_{4,d}$$



Toggweiler 1999

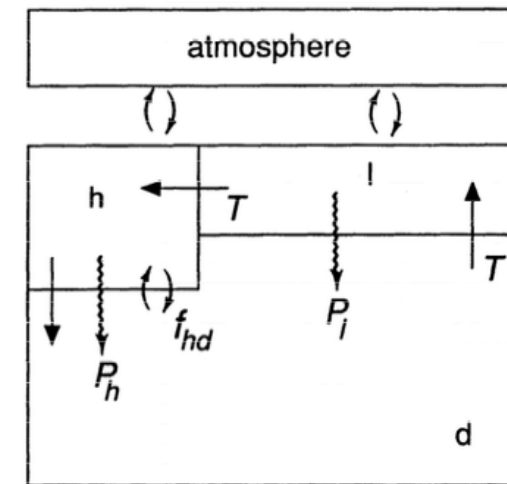
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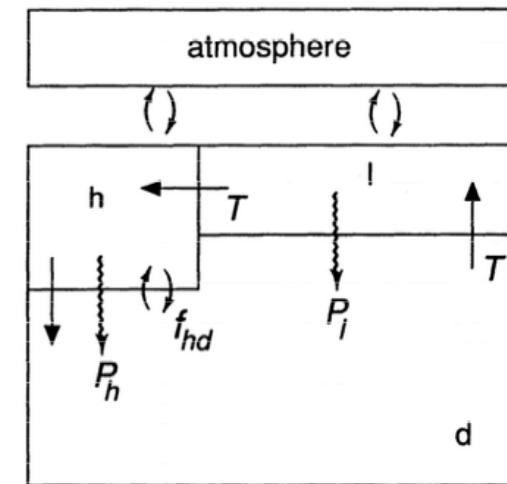
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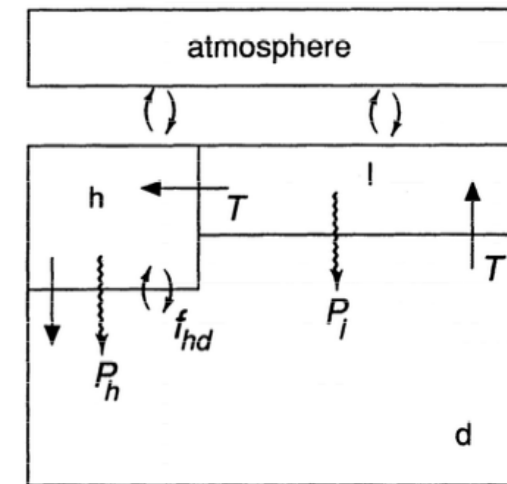
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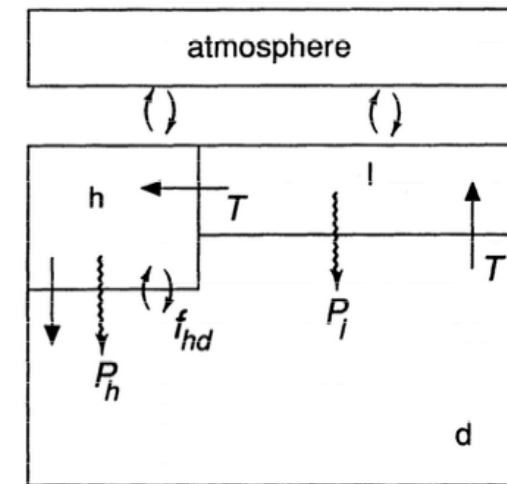
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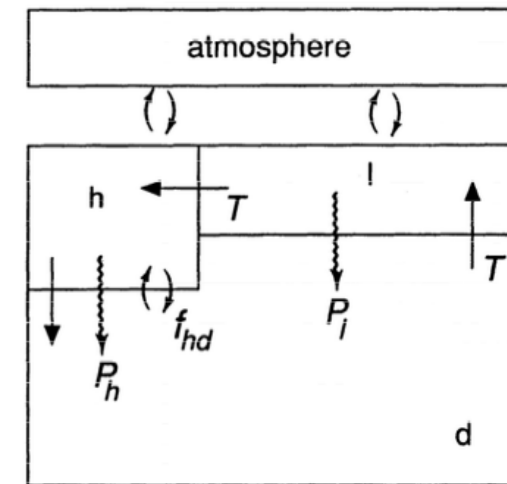
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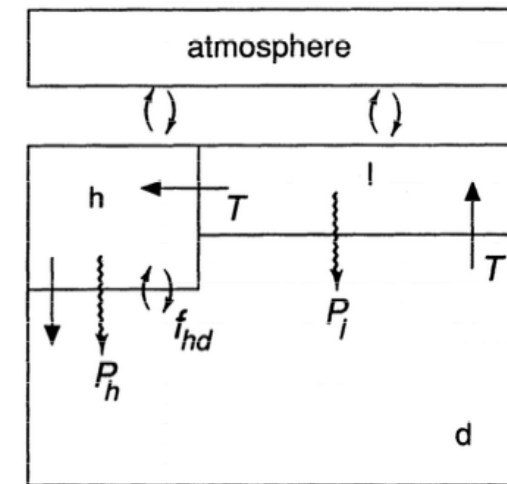
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Turns out this explanation contradicts proxy evidence of SO productivity, back to square 1

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- 100 kyr time scale
- Saw-tooth structure: long glaciations (~90,000 yr), short deglaciations (10,000 yr)
- The transition from 41 kyr to 100 kyr glacial cycles ~800 kyr ago
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Glacial cycles, summary

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What we can say:

- Likely a self-sustained relaxation oscillation that would have existed without Milankovitch and CO₂ variations
- Phase locked by obliquity (3:1) of precession (4:1 or 5:1)
- 41 kyr oscillation likely driven by integrated summer insolation
- Amplified by CO₂ variations; CO₂ varies due to solubility & biological pumps

Mechanism for glacial cycles is still unresolved, although this is the largest climate variability signal over the past 1 Myr; Mechanism for CO₂ variations also still not clear

The End