CHAPTER 10

Structural Dynamics

10.1 INTRODUCTION

In addition to static analyses, the finite element method is a powerful tool for analyzing the dynamic response of structures. As illustrated in Chapter 7, the finite element method in combination with the finite difference method can be used to examine the transient response of heat transfer situations. A similar approach can be used to analyze the transient dynamic response of mechanical structures. However, in the analysis of structures, an additional tool is available. The tool, known as *modal analysis,* has its basis in the fact that every mechanical structure exhibits natural modes of vibration (dynamic response) and these modes can be readily computed given the elastic and inertia characteristics of the structure.

In this chapter, we introduce the concept of natural modes of vibration via the simple harmonic oscillator system. Using the finite element concepts developed in earlier chapters, the simple harmonic oscillator is represented as a finite element system and the basic ideas of natural frequency and natural mode are introduced. The single degree of freedom simple harmonic oscillator is then extended to multiple degrees of freedom, to illustrate the existence of multiple natural frequencies and vibration modes. From this basis, we proceed to more general dynamic analyses using the finite element method.

10.2 THE SIMPLE HARMONIC OSCILLATOR

The so-called simple harmonic oscillator is a combination of a linear elastic spring having free length L and a concentrated mass as shown in Figure 10.1a. The mass of the spring is considered negligible. The system is assumed to be subjected to gravity in the vertical direction, and the upper end of the spring is attached to a rigid support. With the system in equilibrium as in Figure 10.1b, the

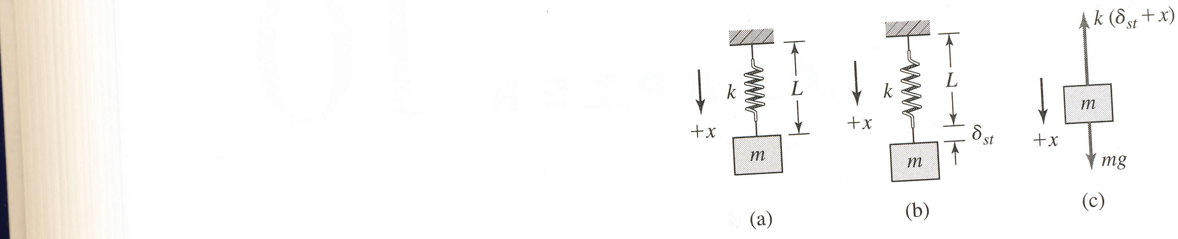

Figure 10.1
(a) Simple harmonic oscillator. (b) Static equilibrium position. (c) Free-body diagram for arbitrary position x.

gravitational force is in equilibrium with the spring force so

$$\sum F_x = 0 = mg - k\delta_{st} \tag{10.1}$$

where δ_{st} is the equilibrium elongation of the spring and x is measured positive downward from the equilibrium position; that is, when $x = 0$, the system is at its equilibrium position.

If, by some action, the mass is displaced from its equilibrium position, the force system becomes unbalanced, as shown by the free-body diagram of Figure 10.1c. We must apply Newton's second law to obtain

$$\sum F_x = ma_x = m\frac{d^2x}{dt^2} = mg - k(\delta_{st} + x) \tag{10.2}$$

Incorporating the equilibrium condition expressed by Equation 10.1, Equation 10.2 becomes

$$m\frac{d^2x}{dt^2} + kx = 0 \tag{10.3}$$

Equation 10.3 is a second-order, linear, ordinary differential equation with constant coefficients. (And physically, we assume that the coefficients m and k are positive.) Equation 10.3 is most-often expressed in the form

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = \frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{10.4}$$

The general solution for Equation 10.4 is

$$x(t) = A \sin \omega t + B \cos \omega t \tag{10.5}$$

where A and B are the constants of integration. Recall that the solution of a second-order differential equation requires the specification of two constants to determine the solution to a specific problem. When the differential equation describes the time response of a mechanical system, the constants of integration are most-often called the *initial conditions*.

Equation 10.5 shows that the variation of displacement of the mass as a function of time is periodic. Using basic trigonometric identities, Equation 10.5 can be equivalently expressed as

$$x(t) = C\sin(\omega t + \phi) \tag{10.6}$$

where the constants A and B have been replaced by constants of integration C and ϕ. Per Equation 10.6, the mass oscillates sinusoidally at *circular frequency* ω and with constant *amplitude* C. Phase angle ϕ is indicative of position at time 0 since $x(0) = C\sin\phi$. Also, note that, since $x(t)$ is measured about the equilibrium position, the oscillation occurs about that position. The circular frequency is

$$\omega = \sqrt{\frac{k}{m}}\ \text{rad/sec} \tag{10.7}$$

and is a constant value determined by the physical characteristics of the system. In this simple case, the *natural circular frequency,* as it is often called, depends on the spring constant and mass only. Therefore, if the mass is displaced from the equilibrium position and released, the oscillatory motion occurs at a constant frequency determined by the physical parameters of the system. In the case described, the oscillatory motion is described as *free vibration,* since the system is free of all external forces excepting gravitational attraction.

Next, we consider the simple harmonic oscillator in the finite element context. From Chapter 2, the stiffness matrix of the spring is

$$[k^{(e)}] = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{10.8}$$

and the equilibrium equations for the element are

$$k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{10.9}$$

which is identical to Equation 2.4. However, the spring element is not in static equilibrium, so we must examine the nodal forces in detail.

Figure 10.2 shows free-body diagrams of the spring element and mass, respectively. The free-body diagrams depict snapshots in time when the system is in motion and, hence, are dynamic free-body diagrams. As the mass of the spring is considered negligible, Equation 10.9 is valid for the spring element. For the mass, we have

$$\sum F_x = ma_x = m\frac{d^2u_2}{dt^2} = mg - f_2 \tag{10.10}$$

from which the force on node 2 is

$$f_2 = mg - m\frac{d^2u_2}{dt^2} \tag{10.11}$$

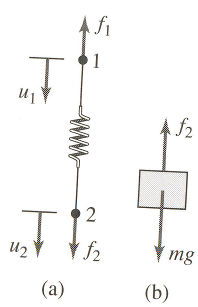

Figure 10.2 Free-body diagrams of (a) a spring and (b) a mass, when treated as parts of a finite element system.

Substituting for f_2 in Equation 10.9 gives

$$k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ mg - m\ddot{u}_2 \end{Bmatrix} \tag{10.12}$$

where $\ddot{u}_2 = \mathrm{d}^2u_2/\mathrm{d}t^2$. The dynamic effect of the inertia of the attached mass is shown in the second of the two equations represented by Equation 10.12. Equation 10.12 can also be expressed as

$$\begin{bmatrix} 0 & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ mg \end{Bmatrix} \tag{10.13}$$

where we have introduced the mass matrix

$$[m] = \begin{bmatrix} 0 & 0 \\ 0 & m \end{bmatrix} \tag{10.14}$$

and the nodal acceleration matrix

$$\{\ddot{u}\} = \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} \tag{10.15}$$

For the simple harmonic oscillator of Figure 10.1, we have the constraint (boundary) condition $u_1 = 0$, so the first of Equation 10.13 becomes simply $-ku_2 = f_1$, while the second equation is

$$m\ddot{u}_2 + ku_2 = mg \tag{10.16}$$

Note that Equation 10.16 is *not* the same as Equation 10.3. Do the two equations represent the same physical phenomenon? To show that the answer is yes, we solve Equation 10.16 and compare the results with the solution given in Equation 10.6.

Recalling that the solution of any differential equation is the sum of a homogeneous (complementary) solution and a particular solution, both solutions must be obtained for Equation 10.16, since the equation is not homogeneous (i.e., the right-hand side is nonzero). Setting the right-hand side to zero, the form of the homogeneous equation is the same as that of Equation 10.3, so by analogy, the homogeneous solution is

$$u_{2h}(t) = C\sin(\omega t + \phi) \tag{10.17}$$

where ω, C, and ϕ are as previously defined. The particular solution must satisfy Equation 10.16 exactly for all values of time. As the right-hand side is constant, the particular solution must also be constant; hence,

$$u_{2p}(t) = \frac{mg}{k} = \delta_{st} \tag{10.18}$$

which represents the static equilibrium solution per Equation 10.1. The complete solution is then

$$u_2(t) = u_{2h}(t) + u_{2p}(t) = \delta_{st} + C\sin(\omega t + \phi) \tag{10.19}$$

Equation 10.19 represents a sinusoidal oscillation around the equilibrium position and is, therefore, the same as the solution given in Equation 10.6. Given the displacement of node 2, the reaction force at node 1 is obtained via the constraint equation as

$$f_1 = -ku_2(t) = -k(\delta_{st} + C \sin(\omega t + \phi)) \tag{10.20}$$

Amplitude C and phase angle ϕ are determined by application of the initial conditions, as illustrated in the following example.

EXAMPLE 10.1

A simple harmonic oscillator has $k = 25$ lb/in. and $mg = 20$ lb. The mass is displaced downward a distance of 1.5 in. from the equilibrium position. The mass is released from that position with zero initial velocity at $t = 0$. Determine (a) the natural circular frequency, (b) the amplitude of the oscillatory motion, and (c) the phase angle of the oscillatory motion.

■ **Solution**

The natural circular frequency is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{25}{20/386.4}} = 21.98 \text{ rad/sec}$$

where, for consistency of units, the mass is obtained from the weight using $g = 386.4$ in./s^2.

The given initial conditions are

$$u_2(t=0) = \delta_{st} + 1.5 \text{ in.} \qquad \dot{u}_2(t=0) = 0 \text{ in./sec}$$

and the static deflection is $\delta_{st} = W/k = 20/25 = 0.8$ in. Therefore, we have $u_2(0) =$ 2.3 in. The motion of node 2 (hence, the mass) is then given by Equation 10.19 as

$$u_2(t) = 0.8 + C \sin(21.98t + \phi) \text{ in.}$$

and the velocity is

$$\dot{u}_2(t) = \frac{du_2}{dt} = 21.98C \cos(21.98t + \phi) \text{ in./sec}$$

Applying the initial conditions results in the equations

$$u_2(t=0) = 2.3 = 0.8 + C \sin\phi$$
$$\dot{u}_2(t) = 0 = 21.98C \cos\phi$$

The initial velocity equation is satisfied by $C = 0$ or $\phi = \pi/2$. If the former is true, the initial displacement equation cannot be satisfied, so we conclude that $\phi = \pi/2$. Substituting into the displacement equation then gives the amplitude C as 1.5 in. The complete motion solution is

$$u_2(t) = 0.8 + 1.5 \sin\left(21.98t + \frac{\pi}{2}\right) = 0.8 + 1.5 \cos(21.98t) \text{ in.}$$

indicating that the mass oscillates 1.5 in. above and below the static equilibrium position continuously in time and completes one cycle every $2\pi/21.98$ sec. Therefore, the cyclic frequency is

$$f = \frac{\omega}{2\pi} = \frac{21.98}{2\pi} = 3.5 \text{ cycles/sec (Hz)}$$

The cyclic frequency is often simply referred to as the *natural frequency.* The time required to complete one cycle of motion is known as the *period* of oscillation, given by

$$\tau = \frac{1}{f} = \frac{1}{3.5} = 0.286 \text{ sec}$$

10.2.1 Forced Vibration

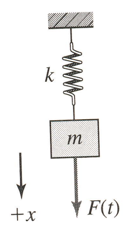

Figure 10.3 Simple harmonic oscillator subjected to external force $F(t)$.

Figure 10.3 shows a simple harmonic oscillator in which the mass is acted on by a time-varying external force $F(t)$. The resulting motion is known as *forced vibration,* owing to the presence of the external forcing function. As the only difference in the applicable free-body diagrams is the external force acting on the mass, the finite element form of the system equations can be written directly from Equation 10.13 as

$$\begin{bmatrix} 0 & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ mg + F(t) \end{Bmatrix} \tag{10.21}$$

While the constraint equation for the reaction force at node 1 is unchanged, the differential equation for the motion of node 2 is now

$$m\ddot{u}_2 + ku_2 = mg + F(t) \tag{10.22}$$

The complete solution for Equation 10.22 is the sum of the homogeneous solution and two particular solutions, since two nonzero terms are on the right-hand side. As we already obtained the homogeneous solution and the particular solution for the mg term, we focus on the particular solution for the external force. The particular solution of interest must satisfy

$$m\ddot{u}_2 + ku_2 = F(t) \tag{10.23}$$

exactly for all values of time. Dividing by the mass, we obtain

$$\ddot{u}_2 + \omega^2 u_2 = \frac{F(t)}{m} \tag{10.24}$$

where $\omega^2 = k/m$ is the square of the natural circular frequency. Of particular importance in structural dynamic analysis is the case when external forcing functions exhibit sinusoidal variation in time, since such forces are quite common. Therefore, we consider the case in which

$$F(t) = F_0 \sin \omega_f t \tag{10.25}$$

where F_0 is the amplitude or maximum value of the force and ω_f is the circular frequency of the forcing function, or *forcing frequency* for short. Equation 10.24 becomes

$$\ddot{u}_2 + \omega^2 u_2 = \frac{F_0}{m} \sin \omega_f t \tag{10.26}$$

To satisfy Equation 10.24 exactly for all values of time, the terms on the left must contain a sine function identical to the sine term on the right-hand side. Since the second derivative of the sine function is another sine function, we assume a solution in the form $u_2(t) = U \sin \omega_f t$, where U is a constant to be determined. Differentiating twice and substituting, Equation 10.26 becomes

$$-U\omega_f^2 \sin \omega_f t + U\omega^2 \sin \omega_f t = \frac{F_0}{m} \sin \omega_f t \tag{10.27}$$

from which

$$U = \frac{F_0/m}{\omega^2 - \omega_f^2} \tag{10.28}$$

The particular solution representing response of the simple harmonic oscillator to a sinusoidally varying force is then

$$u_2(t) = \frac{F_0/m}{\omega^2 - \omega_f^2} \sin \omega_f t \tag{10.29}$$

The motion represented by Equation 10.29 is most often simply called the *forced response* and exhibits two important characteristics: (1) the frequency of the forced response is the same as the frequency of the forcing function, and (2) if the circular frequency of the forcing function is very near the natural circular frequency of the system, the denominator in Equation 10.29 becomes very small. The latter is an extremely important observation, as the result is large amplitude of motion. In the case $\omega_f = \omega$, Equation 10.29 indicates an infinite amplitude. This condition is known as *resonance,* and for this reason, the natural circular frequency of the system is often called the *resonant frequency*. Mathematically, Equation 10.29 is not a valid solution for the resonant condition (Problem 10.5); however, the correct solution for the resonant condition nevertheless exhibits unbounded amplitude growth with time.

The simple harmonic oscillator just modeled contains no device for energy dissipation (*damping*). Consequently, the free vibration solution, Equation 10.20, represents motion that continues without end. Physically, such motion is not possible, since all systems contain some type of dissipation mechanism, such as internal or external friction, air resistance, or devices specifically designed for the purpose. Similarly, the infinite amplitude indicated for the resonant condition cannot be attained by a real system because of the presence of damping. However, relatively large, yet bounded, amplitudes occur at or near the resonant frequency. Hence, the resonant condition is to be avoided if at all possible. As is subsequently shown, physical systems actually exhibit multiple natural frequencies, so multiple resonant conditions exist.

10.3 MULTIPLE DEGREES-OF-FREEDOM SYSTEMS

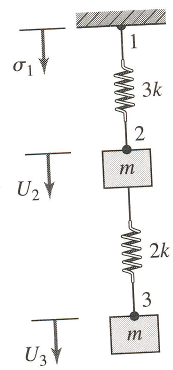

Figure 10.4 A spring-mass system exhibiting 2 degrees of freedom.

Figure 10.4 shows a system of two spring elements having concentrated masses attached at nodes 2 and 3 in the global coordinate system. As in previous examples, the system is subjected to gravity and the upper spring is attached to a rigid support at node 1. Of interest here is the dynamic response of the system of two springs and two masses when the equilibrium condition is disturbed by some external influence and then free to oscillate without external force. We could take the Newtonian mechanics approach by drawing the appropriate free-body diagrams and applying Newton's second law of motion to obtain the governing equations. Instead, we take the finite element approach. By now, the procedure of assembling the system stiffness matrix should be routine. Following the procedure, we obtain

$$[K] = \begin{bmatrix} 3k & -3k & 0 \\ -3k & 5k & -2k \\ 0 & -2k & 2k \end{bmatrix} \tag{10.30}$$

as the system stiffness matrix. But what of the mass/inertia matrix? As the masses are concentrated at element nodes, we define the system mass matrix as

$$[M] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \tag{10.31}$$

The equations of motion can be expressed as

$$[M]\begin{Bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \\ \ddot{U}_3 \end{Bmatrix} + [K]\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ mg \\ mg \end{Bmatrix} \tag{10.32}$$

where R_1 is the dynamic reaction force at node 1.

Invoking the constraint condition $U_1 = 0$, Equation 10.32 become

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{U}_2 \\ \ddot{U}_3 \end{Bmatrix} + \begin{bmatrix} 5k & -2k \\ -2k & 2k \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} mg \\ mg \end{Bmatrix} \tag{10.33}$$

which is a system of two second-order, linear, ordinary differential equations in the two unknown system displacements U_2 and U_3. As the gravitational forces indicated by the forcing function represent the static equilibrium condition, these are neglected and the system of equations rewritten as

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{U}_2 \\ \ddot{U}_3 \end{Bmatrix} + \begin{bmatrix} 5k & -2k \\ -2k & 2k \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{10.34}$$

As a practical matter, most finite element software packages do *not* include the structural weight in an analysis problem. Instead, inclusion of the structural weight is an option that must be selected by the user of the software. Whether to include gravitational effects is a judgment made by the analyst based on the specifics of a given structural geometry and loading.

The system of second-order, linear, ordinary, homogeneous differential equations given by Equation 10.34 represents the free-vibration response of the 2 degrees-of-freedom system of Figure 10.4. As a freely oscillating system, we seek solutions in the form of harmonic motion as

$$\begin{aligned} U_2(t) &= A_2 \sin(\omega t + \phi) \\ U_3(t) &= A_3 \sin(\omega t + \phi) \end{aligned} \tag{10.35}$$

where A_2 and A_3 are the vibration amplitudes of nodes 2 and 3 (the masses attached to nodes 2 and 3); ω is an unknown, assumed harmonic circular frequency of motion; and ϕ is the phase angle of such motion. Taking the second derivatives with respect to time of the assumed solutions and substituting into Equation 10.34 results in

$$-\omega^2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} A_2 \\ A_3 \end{Bmatrix} \sin(\omega t + \phi) + \begin{bmatrix} 5k & -2k \\ -2k & 2k \end{bmatrix} \begin{Bmatrix} A_2 \\ A_3 \end{Bmatrix} \sin(\omega t + \phi) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{10.36}$$

or

$$\begin{bmatrix} 5k - m\omega^2 & -2k \\ -2k & 2k - m\omega^2 \end{bmatrix} \begin{Bmatrix} A_2 \\ A_3 \end{Bmatrix} \sin(\omega t + \phi) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{10.37}$$

Equation 10.37 is a system of two, homogeneous algebraic equations, which must be solved for the vibration amplitudes A_2 and A_3. From linear algebra, a system of homogeneous algebraic equations has nontrivial solutions if and only if the determinant of the coefficient matrix is zero. Therefore, for nontrivial solutions,

$$\begin{vmatrix} 5k - m\omega^2 & -2k \\ -2k & 2k - m\omega^2 \end{vmatrix} = 0 \tag{10.38}$$

which gives

$$(5k - m\omega^2)(2k - m\omega^2) - 4k^2 = 0 \tag{10.39}$$

Equation 10.39 is known as the *characteristic equation* or *frequency equation* of the physical system. As k and m are known positive constants, Equation 10.39 is treated as a quadratic equation in the unknown ω^2 and solved by the quadratic formula to obtain *two* roots

$$\begin{aligned} \omega_1^2 &= \frac{k}{m} \\ \omega_2^2 &= 6\frac{k}{m} \end{aligned} \tag{10.40}$$

or

$$\omega_1 = \sqrt{\frac{k}{m}}$$
$$\omega_2 = \sqrt{6\frac{k}{m}} \tag{10.41}$$

In mathematical rigor, there are four roots, since the negative values corresponding to Equation 10.41 also satisfy the frequency equation. The negative values are rejected because a negative frequency has no physical meaning and use of the negative values in the assumed solution (Equation 10.35) introduces only a phase shift and represents the same motion as that corresponding to the positive root.

The 2 degrees-of-freedom system of Figure 10.4 is found to have two natural circular frequencies of oscillation. As is customary, the numerically smaller of the two is designated as ω_1 and known as the *fundamental* frequency. The task remains to determine the amplitudes A_2 and A_3 in the assumed solution. For this purpose, Equation 10.37 is

$$\begin{bmatrix} 5k - m\omega^2 & -2k \\ -2k & 2k - m\omega^2 \end{bmatrix} \begin{Bmatrix} A_2 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{10.42}$$

As Equation 10.42 is a set of homogeneous equations, we can find no absolute values of the amplitudes. We can, however, obtain information regarding the numerical relations among the amplitudes as follows. If we substitute $\omega^2 = \omega_1^2 = k/m$ into *either* algebraic equation, we obtain $A_3 = 2A_2$, which defines the *amplitude ratio* $A_3/A_2 = 2$ for the first, or fundamental, mode of vibration. That is, if the system oscillates at its fundamental frequency ω_1, the amplitude of oscillation of m_2 is twice that of m_1. (Note that we are unable to calculate the absolute value of either amplitude; only the ratio can be determined. The absolute values depend on the initial conditions of motion, as is subsequently illustrated.) The displacement equations for the fundamental mode are then

$$U_2(t) = A_2^{(1)}\sin(\omega_1 t + \phi_1)$$
$$U_3(t) = A_3^{(1)}\sin(\omega_1 + \phi_1) = 2A_2^{(1)}\sin(\omega_1 t + \phi_1) \tag{10.43}$$

where the superscript on the amplitudes is used to indicate that the displacements correspond to vibration at the fundamental frequency.

Next we substitute the second natural circular frequency $\omega^2 = \omega_2^2 = 6k/m$ into either equation and obtain the relation $A_3 = -0.5A_2$, which defines the second amplitude ratio as $A_3/A_2 = -0.5$. So, in the second natural mode of vibration, the masses move in opposite directions. The displacements corresponding to the second frequency are then

$$U_2(t) = A_2^{(2)}\sin(\omega_2 t + \phi_2)$$
$$U_3(t) = A_3^{(2)}\sin(\omega_2 + \phi_2) = -0.5A_2^{(2)}\sin(\omega_2 t + \phi_2) \tag{10.44}$$

where again the superscript refers to the frequency.

Therefore, the free-vibration response of the 2 degree-of-freedom system is given by

$$U_2(t) = A_2^{(1)}\sin(\omega_1 t + \phi_1) + A_2^{(2)}\sin(\omega_2 t + \phi_2)$$
$$U_3(t) = 2A_2^{(1)}\sin(\omega_1 t + \phi_1) - 0.5A_2^{(2)}\sin(\omega_2 + \phi_2) \tag{10.45}$$

and we note the four unknown constants in the solution; specifically, these are the amplitudes $A_2^{(1)}$, $A_2^{(2)}$ and the phase angles ϕ_1 and ϕ_2. Evaluation of the constants is illustrated in a subsequent example.

Depending on the reader's mathematical background, the analysis of the 2 degree-of-freedom vibration problem may be recognized as an *eigenvalue problem* [1]. The computed natural circular frequencies are the eigenvalues of the problem and the amplitude ratios represent the eigenvectors of the problem. Equation 10.45 represents the response of the system in terms of the natural modes of vibration. Such a solution is often referred to as being obtained by *modal superposition* or simply *modal analysis*. To represent the complete solution for the system, we use the matrix notation

$$\begin{Bmatrix} U_2(t) \\ U_3(t) \end{Bmatrix} = \begin{Bmatrix} A_2^{(1)} \\ 2A_2^{(1)} \end{Bmatrix} \sin(\omega_1 t + \phi_1) + \begin{Bmatrix} A_2^{(2)} \\ -0.5A_2^{(2)} \end{Bmatrix} \sin(\omega_2 t + \phi_2) \tag{10.46}$$

which shows that the modes interact to produce the overall motion of the system.

EXAMPLE 10.2

Given the system of Figure 10.4 with $k = 40$ lb/in. and $mg = W = 20$ lb, determine

(a) The natural frequencies of the system.

(b) The free response, if the initial conditions are

$$U_2(t = 0) = 1 \text{ in.} \qquad U_3(t = 0) = 0.5 \text{ in.} \qquad \dot{U}_2(t = 0) = \dot{U}_3(t = 0) = 0$$

These initial conditions are specified in reference to the equilibrium position of the system, so the computed displacement functions do not include the effect of gravity.

■ Solution

Per Equation 10.41, the natural circular frequencies are

$$\omega_1 = \sqrt{\frac{k}{m}} = \sqrt{\frac{40}{20/g}} = \sqrt{\frac{40(386.4)}{20}} = 27.8 \text{ rad/sec}$$

$$\omega_2 = \sqrt{\frac{6k}{m}} = \sqrt{\frac{6(40)}{20/g}} = \sqrt{\frac{6(40)(386.4)}{20}} = 68.1 \text{ rad/sec}$$

The free-vibration response is given by Equation 10.35 as

$$U_2(t) = A_2^{(1)}\sin(27.8t + \phi_1) + A_2^{(2)}\sin(68.1t + \phi_2)$$
$$U_3(t) = 2A_2^{(1)}\sin(27.8t + \phi_1) - 0.5A_2^{(2)}\sin(68.1t + \phi_2)$$

The amplitudes and phase angles are determined by applying the initial conditions, which are

$$U_2(0) = 1 = A_2^{(1)} \sin\phi_1 + A_2^{(2)} \sin\phi_2$$
$$U_3(0) = 0.5 = 2A_2^{(1)} \sin\phi_1 - 0.5A_2^{(2)} \sin\phi_2$$
$$\dot{U}_2(0) = 0 = 27.8A_2^{(1)} \cos\phi_1 + 68.1A_2^{(2)} \cos\phi_2$$
$$\dot{U}_3(0) = 0 = 2(27.8)A_2^{(1)} \cos\phi_1 - 0.5(68.1)A_2^{(2)} \cos\phi_2$$

The initial conditions produce a system of four algebraic equations in the four unknowns $A_2^{(1)}$, $A_2^{(2)}$, ϕ_1, ϕ_2. Solution of the equations is not trivial, owing to the presence of the trigonometric functions. Letting $P = A_2^{(1)} \sin\phi_1$ and $Q = A_2^{(2)} \sin\phi_2$, the displacement initial condition equations become

$$P + Q = 1$$
$$2P - 0.5Q = 0.5$$

which are readily solved to obtain

$$P = A_2^{(1)} \sin\phi_1 = 0.4 \quad \text{and} \quad Q = A_2^{(2)} \sin\phi_2 = 0.6$$

Similarly, setting $R = A_2^{(1)} \cos\phi_1$ and $S = A_2^{(2)} \sin\phi_2$, the initial velocity equations are

$$27.8R + 68.1S = 0$$
$$2(27.8)R - 0.5(68.1)S = 0$$

representing a homogeneous system in the variables R and S. Nontrivial solutions exist only if the determinant of the coefficient matrix is zero. In this case, the determinant is not zero, as may easily be verified by direct computation. There are no nontrivial solutions; hence, $R = S = 0$. Based on physical argument, the amplitudes cannot be zero, so we must conclude that $\cos\phi_1 = \cos\phi_2 = 0 \Rightarrow \phi_1 = \phi_2 = \pi/2$. It follows that the sine function of the phase angles have unity value; hence, $A_2^{(1)} = 0.4$ and $A_2^{(2)} = 0.6$. Substituting the amplitudes into the general solution form while noting that $\sin(\omega t + \pi/2) = \cos\omega t$, the free-vibration response of each mass is

$$U_2(t) = 0.4\cos 27.8t + 0.6\cos 68.1t$$
$$U_3(t) = 0.8\cos 27.8t - 0.3\cos 68.1t$$

The displacement response of each mass is seen to be a combination of motions corresponding to the natural circular frequencies of the system. Such a phenomenon is characteristic of vibrating structural systems. All the natural modes of vibration participate in the general motion of a structure.

10.3.1 Many-Degrees-of-Freedom Systems

As illustrated by the system of two springs and masses, there are two natural frequencies and two natural modes of vibration. If we extend the analysis to a system of springs and masses having N degrees of freedom, as depicted in Figure 10.5, and apply the assembly procedure for a finite element analysis, the finite element equations are of the form

$$[M]\{\ddot{U}\} + [K]\{U\} = \{0\} \tag{10.47}$$

where $[M]$ is the system mass matrix and $[K]$ is the system stiffness matrix. To determine the natural frequencies and mode shapes of the system's vibration modes, we assume, as in the 1 and 2 degrees-of-freedom cases, that

$$U_i(t) = A_i \sin(\omega t + \phi) \tag{10.48}$$

Substitution of the assumed solution into the system equations leads to the frequency equation

$$|[K] - \omega^2[M]| = 0 \tag{10.49}$$

which is a polynomial of order N in the variable ω^2. The solution of Equation 10.49 results in N natural frequencies ω_j, which, for structural systems, can be shown to be real but not necessarily distinct; that is, repeated roots can occur. As discussed many times, the finite element equations cannot be solved unless boundary conditions are applied so that the equations become inhomogeneous. A similar phenomenon exists when determining the system natural frequencies and mode shapes. If the system is not constrained, rigid body motion is possible and one or more of the computed natural frequencies has a value of zero. A three-dimensional system has six zero-valued natural frequencies, corresponding to rigid body translation in the three coordinate axes and rigid body rotations about the three coordinate axes. Therefore, if improperly constrained, a structural system exhibits repeated zero roots of the frequency equation.

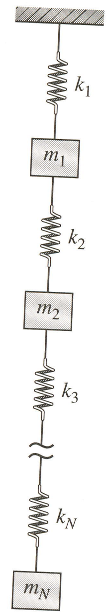

Figure 10. 5 A spring-mass system exhibiting arbitrarily many degrees of freedom.

Assuming that constraints are properly applied, the frequencies resulting from the solution of Equation 10.49 are substituted, one at a time, into Equation 10.47 and the amplitude ratios (eigenvectors) computed for each natural mode of vibration. The general solution for each degree of freedom is then expressed as

$$U_i(t) = \sum_{j=1}^{N} A_i^{(j)} \sin(\omega_j t + \phi_j) \qquad i = 1, N \tag{10.50}$$

illustrating that the displacement of each mass is the sum of contributions from each of the N natural modes. Displacement solutions expressed by Equation 10.50 are said to be obtained by *modal superposition.* We add the independent solutions of the linear differential equations of motion.

EXAMPLE 10.3

Determine the natural frequencies and modal amplitude vectors for the 3 degrees-of-freedom system depicted in Figure 10.6a.

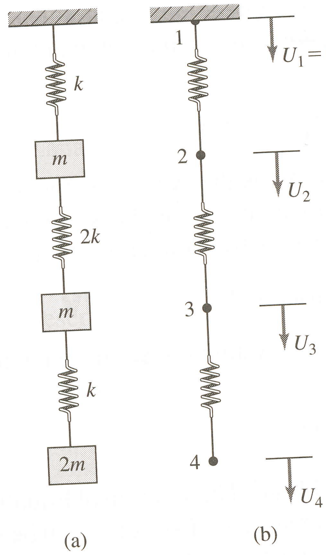

Figure 10.6 System with 3 degrees of freedom for Example 10.3.

■ **Solution**

The finite element model is shown in Figure 10.6b, with node and element numbers as indicated. Assembly of the global stiffness matrix results in

$$[K] = \begin{bmatrix} k & -k & 0 & 0 \\ -k & 3k & -2k & 0 \\ 0 & -2k & 3k & -k \\ 0 & 0 & -k & k \end{bmatrix}$$

Similarly, the assembled global mass matrix is

$$[M] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & 2m \end{bmatrix}$$

Owing to the constraint $U_1 = 0$, we need consider only the last three equations of motion, given by

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 2m \end{bmatrix} \begin{Bmatrix} \ddot{U}_2 \\ \ddot{U}_3 \\ \ddot{U}_4 \end{Bmatrix} + \begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Assuming sinusoidal response as $U_i = A_i \sin(\omega t + \phi)$, $i = 2, 4$ and substituting into the equations of motion leads to the frequency equation

$$\begin{vmatrix} 3k - \omega^2 m & -2k & 0 \\ -2k & 3k - \omega^2 m & -k \\ 0 & -k & k - 2\omega^2 m \end{vmatrix} = 0$$

Expanding the determinant and simplifying gives

$$\omega^6 - 6.5\frac{k}{m}\omega^4 + 7.5\left(\frac{k}{m}\right)^2\omega^2 - \left(\frac{k}{m}\right)^3 = 0$$

which will be treated as a cubic equation in the unknown ω^2. Setting $\omega^2 = C(k/m)$, the frequency equation becomes

$$(C^3 - 6.5C^2 + 7.5C - 1)\left(\frac{k}{m}\right)^3 = 0$$

which has the roots

$$C_1 = 0.1532 \qquad C_2 = 1.2912 \qquad C_3 = 5.0556$$

The corresponding natural circular frequencies are

$$\omega_1 = 0.3914\sqrt{\frac{k}{m}}$$

$$\omega_2 = 1.1363\sqrt{\frac{k}{m}}$$

$$\omega_3 = 2.2485\sqrt{\frac{k}{m}}$$

To obtain the amplitude ratios, we substitute the natural circular frequencies into the amplitude equations one at a time while setting (arbitrarily) $A_2^{(i)} = 1$, $i = 1, 2, 3$ and solve for the amplitudes $A_3^{(i)}$ and $A_4^{(i)}$. Using ω_1 results in

$$\begin{aligned}(3k - \omega_1^2 m)A_2^{(1)} - 2kA_3^{(1)} &= 0\\ -2kA_2^{(1)} + (3k - \omega_1^2 m)A_3^{(1)} - kA_4^{(1)} &= 0\\ -kA_3^{(1)} + (k - 2\omega_1^2 m)A_4^{(1)} &= 0\end{aligned}$$

Substituting $\omega_1 = 0.3914\sqrt{k/m}$, we obtain

$$\begin{aligned}2.847A_2^{(1)} - 2A_3^{(1)} &= 0\\ -2A_2^{(1)} + 2.847A_3^{(1)} - A_4^{(1)} &= 0\\ -A_3^{(1)} + 0.694A_4^{(1)} &= 0\end{aligned}$$

As discussed, the amplitude equations are homogeneous; explicit solutions cannot be obtained. We can, however, determine the amplitude ratios by setting $A_2^{(1)} = 1$ to obtain

$$A_3^{(1)} = 1.4235$$

$$A_4^{(1)} = 2.0511$$

The amplitude vector corresponding to the fundamental mode ω_1 is then represented as

$$\{A^{(1)}\} = A_2^{(1)}\begin{Bmatrix}1\\1.4325\\2.0511\end{Bmatrix}$$

and this is the *eigenvector* corresponding to the *eigenvalue* ω_1. Proceeding identically with the values for the other two frequencies, ω_2 and ω_3, the resulting amplitude vectors are

$$\{A^{(2)}\} = A_2^{(2)}\begin{Bmatrix} 1 \\ 0.8544 \\ -0.5399 \end{Bmatrix}$$

$$\{A^{(3)}\} = A_2^{(3)}\begin{Bmatrix} 1 \\ -1.0279 \\ 0.1128 \end{Bmatrix}$$

This example illustrates that an N degree-of-freedom system exhibits N natural modes of vibration defined by N natural circular frequencies and the corresponding N amplitude vectors (mode shapes). While the examples deal with discrete spring-mass systems, where the motions of the masses are easily visualized as recognizable events, structural systems modeled via finite elements exhibit N natural frequencies and N mode shapes, where N is the number of degrees of freedom (displacements in structural systems) represented by the finite element model. Accuracy of the computed frequencies as well as use of the natural modes of vibration to examine response to external forces is delineated in following sections.

10.4 BAR ELEMENTS: CONSISTENT MASS MATRIX

In the preceding discussions of spring-mass systems, the mass (inertia) matrix in each case is a lumped (diagonal) matrix, since each mass is directly attached to an element node. In these simple cases, we neglect the mass of the spring elements in comparison to the concentrated masses. In the general case of solid structures, the mass is distributed geometrically throughout the structure and the inertia properties of the structure depend directly on the mass distribution. To illustrate the effects of distributed mass, we first consider longitudinal (axial) vibration of the bar element of Chapter 2.

The bar element shown in Figure 10.7a is the same as the bar element introduced in Chapter 2 with the very important difference that displacements and applied forces are now assumed to be time dependent, as indicated. The free-body diagram of a differential element of length dx is shown in Figure 10.7b, where cross-sectional area A is assumed constant. Applying Newton's second law to the differential element gives

$$\left(\sigma + \frac{\partial \sigma}{\partial x}\,\mathrm{d}x\right)A - \sigma A = (\rho A\,\mathrm{d}x)\frac{\partial^2 u}{\partial t^2} \tag{10.51}$$

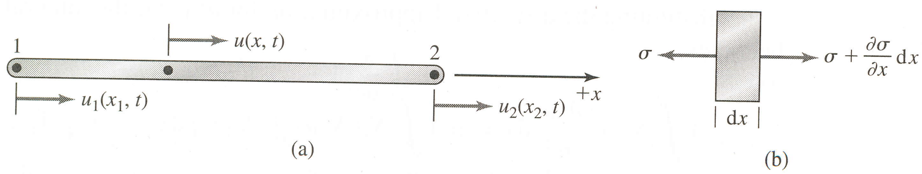

Figure 10.7
(a) Bar element exhibiting time-dependent displacement. (b) Free-body diagram of a differential element.

where ρ is density of the bar material. Note the use of partial derivative operators, since displacement is now considered to depend on both position and time. Substituting the stress-strain relation $\sigma = E\varepsilon = E(\partial u/\partial x)$, Equation 10.51 becomes

$$E\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial t^2} \tag{10.52}$$

Equation 10.52 is the *one-dimensional wave equation,* the governing equation for propagation of elastic displacement waves in the axial bar.

In the dynamic case, the axial displacement is discretized as

$$u(x, t) = N_1(x)u_1(t) + N_2(x)u_2(t) \tag{10.53}$$

where the nodal displacements are now expressed explicitly as time dependent, but the interpolation functions remain dependent only on the spatial variable. Consequently, the interpolation functions are identical to those used previously for equilibrium situations involving the bar element: $N_1(x) = 1 - (x/L)$ and $N_2(x) = x/L$. Application of Galerkin's method to Equation 10.52 in analogy to Equation 5.29 yields the residual equations as

$$\int_0^L N_i(x)\left(E\frac{\partial^2 u}{\partial x^2} - \rho\frac{\partial^2 u}{\partial t^2}\right)A\,\mathrm{d}x = 0 \qquad i = 1, 2 \tag{10.54}$$

Assuming constant material properties, Equation 10.54 can be written as

$$\rho A\int_0^L N_i(x)\frac{\partial^2 u}{\partial t^2}\,\mathrm{d}x = AE\int_0^L N_i(x)\frac{\partial^2 u}{\partial x^2}\,\mathrm{d}x \qquad i = 1, 2 \tag{10.55}$$

Mathematical treatment of the right-hand side of Equation 10.55 is identical to that presented in Chapter 5 and is not repeated here, other than to recall that the result of the integration and combination of the two residual equations in matrix form is

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \Rightarrow [k]\{u\} = \{f\} \tag{10.56}$$

Substituting the discretized approximation for $u(x, t)$, the integral on the left becomes

$$\rho A \int_0^L N_i(x) \frac{\partial^2 u}{\partial t^2} \, \mathrm{d}x = \rho A \int_0^L N_i (N_1 \ddot{u}_1 + N_2 \ddot{u}_2) \, \mathrm{d}x \qquad i = 1, 2 \tag{10.57}$$

where the double-dot notation indicates differentiation with respect to time. The two equations represented by Equation 10.57 are written in matrix form as

$$\rho A \int_0^L \begin{bmatrix} N_1^2 & N_1 N_2 \\ N_1 N_2 & N_2^2 \end{bmatrix} \mathrm{d}x \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} = \frac{\rho A L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} = [m]\{\ddot{u}\} \tag{10.58}$$

and the reader is urged to confirm the result by performing the indicated integrations. Also note that the mass matrix is symmetric but not singular. Equation 10.58 defines the *consistent* mass matrix for the bar element. The term *consistent* is used because the interpolation functions used in formulating the mass matrix are the same as (consistent with) those used to describe the spatial variation of displacement. Combining Equations 10.56 and 10.58 per Equation 10.55, we obtain the dynamic finite element equations for a bar element as

$$\frac{\rho A L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{10.59}$$

or

$$[m]\{\ddot{u}\} + [k]\{u\} = \{f\} \tag{10.60}$$

and we note that $\rho A L = m$ is the total mass of the element. (Why is the sign of the second term positive?)

Given the governing equations, let us now determine the natural frequencies of a bar element in axial vibration. Per the foregoing discussion of free vibration, we set the nodal force vector to zero and write the frequency equation as

$$|[k] - \omega^2 [m]| = 0 \tag{10.61}$$

to obtain

$$\begin{vmatrix} k - \omega^2 \dfrac{m}{3} & -\left(k + \omega^2 \dfrac{m}{6}\right) \\ -\left(k + \omega^2 \dfrac{m}{6}\right) & k - \omega^2 \dfrac{m}{3} \end{vmatrix} = 0 \tag{10.62}$$

Expanding Equation 10.62 results in a quadratic equation in ω^2

$$\left(k - \omega^2 \frac{m}{3}\right)^2 - \left(k + \omega^2 \frac{m}{6}\right)^2 = 0 \tag{10.63}$$

or

$$\omega^2\left(\omega^2 - 12\frac{k}{m}\right) = 0 \tag{10.64}$$

Equation 10.64 has roots $\omega^2 = 0$ and $\omega^2 = 12k/m$. The zero root arises because we specify no constraint on the element; hence, rigid body motion is possible and represented by the zero-valued natural circular frequency. The nonzero natural circular frequency corresponds to axial displacement waves in the bar, which could occur, for example, if the free bar were subjected to an axial impulse at one end. In such a case, rigid body motion would occur but axial vibration would simultaneously occur with circular frequency $\omega_1 = \sqrt{12k/m} = (3.46/L)\sqrt{E/\rho}$. The following example illustrates determination of natural circular frequencies for a constrained bar.

EXAMPLE 10.4

Using two equal-length finite elements, determine the natural circular frequencies of the solid circular shaft fixed at one end shown in Figure 10.8a.

■ Solution

The elements and node numbers are shown in Figure 10.8b. The characteristic stiffness of each element is

$$k = \frac{AE}{L/2} = \frac{2AE}{L}$$

so that the element stiffness matrices are

$$\left[k^{(1)}\right] = \left[k^{(2)}\right] = \frac{2AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The mass of each element is

$$m = \frac{\rho AL}{2}$$

and the element consistent mass matrices are

$$\left[m^{(1)}\right] = \left[m^{(2)}\right] = \frac{\rho AL}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

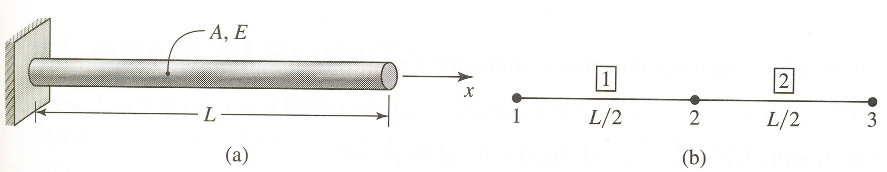

Figure 10.8
(a) Circular shaft of Example 10.4. (b) Model using two bar elements.

Following the direct assembly procedure, the global stiffness matrix is

$$[K] = \frac{2AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

and the global consistent mass matrix is

$$[M] = \frac{\rho AL}{12}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

The global equations of motion are then

$$\frac{\rho AL}{12}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \\ \ddot{U}_3 \end{Bmatrix} + \frac{2AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Applying the constraint condition $U_1 = 0$, we have

$$\frac{\rho AL}{12}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{U}_2 \\ \ddot{U}_3 \end{Bmatrix} + \frac{2AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

as the homogeneous equations governing free vibration. For convenience, the last equation is rewritten as

$$\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{U}_2 \\ \ddot{U}_3 \end{Bmatrix} + \frac{24E}{\rho L^2}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Assuming sinusoidal responses

$$U_2 = A_2 \sin(\omega t + \phi) \qquad U_3 = A_3 \sin(\omega t + \phi)$$

differentiating twice and substituting results in

$$-\omega^2\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} A_2 \\ A_3 \end{Bmatrix}\sin(\omega t + \phi) + \frac{24E}{\rho L^2}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} A_2 \\ A_3 \end{Bmatrix}\sin(\omega t + \phi) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Again, we obtain a set of homogeneous algebraic equations that have nontrivial solutions only if the determinant of the coefficient matrix is zero. Letting $\lambda = 24E/\rho L^2$, the frequency equation is given by the determinant

$$\begin{vmatrix} 2\lambda - 4\omega^2 & -\lambda - \omega^2 \\ -\lambda - \omega^2 & \lambda - 2\omega^2 \end{vmatrix} = 0$$

which, when expanded and simplified, is

$$7\omega^4 - 10\lambda\omega^2 + \lambda^2 = 0$$

Treating the frequency equation as a quadratic in ω^2, the roots are obtained as

$$\omega_1^2 = 0.1082\lambda \qquad \omega_2^2 = 1.3204\lambda$$

Substituting for λ, the natural circular frequencies are

$$\omega_1 = \frac{1.611}{L}\sqrt{\frac{E}{\rho}} \qquad \omega_2 = \frac{5.629}{L}\sqrt{\frac{E}{\rho}} \text{ rad/sec}$$

For comparison purposes, we note that the exact solution [2] for the natural circular frequencies of a bar in axial vibration yields the fundamental natural circular frequency as $1.571/L\sqrt{E/\rho}$ and the second frequency as $4.712/L\sqrt{E/\rho}$. Therefore, the error for the first computed frequency is about 2.5 percent, while the error in the second frequency is about 19 percent.

It is also informative to note (see Problem 10.12) that, if the lumped mass matrix approach is used for this example, we obtain

$$\omega_1 = \frac{1.531}{L}\sqrt{\frac{E}{\rho}} \qquad \omega_2 = \frac{3.696}{L}\sqrt{\frac{E}{\rho}} \text{ rad/sec}$$

The solution for Example 10.4 yielded two natural circular frequencies for free axial vibration of a bar fixed at one end. Such a bar has an infinite number of natural frequencies, like any element or structure having continuously distributed mass. In finite element modeling, the partial differential equations governing motion of continuous systems are discretized into a finite number of algebraic equations for approximate solutions. Hence, the number of frequencies obtainable via a finite element approach is limited by the discretization inherent to the finite element model.

The inertia characteristics of a bar element can also be represented by a lumped mass matrix, similar to the approach used in the spring-mass examples earlier in this chapter. In the lumped matrix approach, half the total mass of the element is assumed to be concentrated at each node and the connecting material is treated as a massless spring with axial stiffness. The lumped mass matrix for a bar element is then

$$[m] = \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{10.65}$$

Use of lumped mass matrices offers computational advantages. Since the element mass matrix is diagonal, assembled global mass matrices also are diagonal. On the other hand, although more computationally difficult in use, consistent mass matrices can be proven to provide upper bounds for the natural circular frequencies [3]. No such proof exists for lumped matrices. Nevertheless, lumped mass matrices are often used, particularly with bar and beam elements, to obtain reasonably accurate predictions of dynamic response.

10.5 BEAM ELEMENTS

We now develop the mass matrix for a beam element in flexural vibration. First, the consistent mass matrix is obtained using an approach analogous to that for the bar element in the previous section. Figure 10.9 depicts a differential element of a beam in flexure under the assumption that the applied loads are time dependent. As the situation is otherwise the same as that of Figure 5.3 except for the use of

Figure 10.9 Differential element of a beam subjected to time-dependent loading.

partial derivatives, we apply Newton's second law of motion to the differential element in the y direction to obtain

$$\sum F_y = ma_y \Rightarrow V + \frac{\partial V}{\partial x}\,dx - V - q(x,t)\,dx = (\rho A\,dx)\frac{\partial^2 v}{\partial t^2} \tag{10.66}$$

where ρ is the material density and A is the cross-sectional area of the element. The quantity ρA represents mass per unit length in the x direction. Equation 10.66 simplifies to

$$\frac{\partial V}{\partial x} - q(x,t) = \rho A \frac{\partial^2 v}{\partial t^2} \tag{10.67}$$

As we are dealing with the small deflection theory of beam flexure, beam slopes, therefore rotations, are small. Therefore, we neglect the rotational inertia of the differential beam element and apply the moment equilibrium equation. The result is identical to that of Equation 5.37, repeated here as

$$\frac{\partial M}{\partial x} = -V \tag{10.68}$$

Substituting the moment-shear relation into Equation 10.67 gives

$$-\frac{\partial^2 M}{\partial x^2} - q(x,t) = \rho A \frac{\partial^2 v}{\partial t^2} \tag{10.69}$$

Finally, the flexure formula

$$M = EI_z \frac{\partial^2 v}{\partial x^2} \tag{10.70}$$

is substituted into Equation 10.69 to obtain the governing equation for dynamic beam deflection as

$$-\frac{\partial^2}{\partial x^2}\left(EI_z \frac{\partial^2 v}{\partial x^2}\right) - q(x,t) = \rho A \frac{\partial^2 v}{\partial t^2} \tag{10.71}$$

Under the assumptions of constant elastic modulus E and moment of inertia I_z, the governing equation becomes

$$\rho A \frac{\partial^2 v}{\partial t^2} + EI_z \frac{\partial^4 v}{\partial x^4} = -q(x,t) \tag{10.72}$$

As in the case of the bar element, transverse beam deflection is discretized using the same interpolation functions previously developed for the beam function. Now, however, the nodal displacements are assumed to be time dependent. Hence,

$$v(x,t) = N_1(x)v_1(t) + N_2(x)\theta_1(t) + N_3(x)v_2(t) + N_4(x)\theta_2(t) \qquad (10.73)$$

and the interpolation functions are as given in Equation 4.26 or 4.29. Application of Galerkin's method to Equation 10.72 for a finite element of length L results in the residual equations

$$\int_0^L N_i(x)\left(\rho A \frac{\partial^2 v}{\partial t^2} + EI_z \frac{\partial^4 v}{\partial x^4} + q\right) = 0 \qquad i = 1, 4 \qquad (10.74)$$

As the last two terms of the integrand are the same as treated in Equation 5.42, development of the stiffness matrix and nodal force vector are not repeated here. Instead, we focus on the first term of the integrand, which represents the terms of the mass matrix.

For each of the four equations represented by Equation 10.74, the first integral term becomes

$$\rho A \int_0^L N_i(N_1\ddot{v}_1 + N_2\ddot{\theta}_1 + N_3\ddot{v}_2 + N_4\ddot{\theta}_2)\,dx = \rho A \int_0^L N_i[N]\,dx \begin{Bmatrix} \ddot{v}_1 \\ \ddot{\theta}_1 \\ \ddot{v}_2 \\ \ddot{\theta}_2 \end{Bmatrix} \qquad i = 1, 4 \qquad (10.75)$$

and, when all four equations are expressed in matrix form, the inertia terms become

$$\rho A \int_0^L [N]^T[N]\,dx \begin{Bmatrix} \ddot{v}_1 \\ \ddot{\theta}_1 \\ \ddot{v}_2 \\ \ddot{\theta}_2 \end{Bmatrix} = \left[m^{(e)}\right] \begin{Bmatrix} \ddot{v}_1 \\ \ddot{\theta}_1 \\ \ddot{v}_2 \\ \ddot{\theta}_2 \end{Bmatrix} \qquad (10.76)$$

The consistent mass matrix for a two-dimensional beam element is given by

$$\left[m^{(e)}\right] = \rho A \int_0^L [N]^T[N]\,dx \qquad (10.77)$$

Substitution for the interpolation functions and performing the required integrations gives the mass matrix as

$$\left[m^{(e)}\right] = \frac{\rho A L}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix} \qquad (10.78)$$

and it is to be noted that we have assumed constant cross-sectional area in this development.

Combining the mass matrix with previously obtained results for the stiffness matrix and force vector, the finite element equations of motion for a beam element are

$$[m^{(e)}]\begin{Bmatrix}\ddot{v}_1\\ \ddot{\theta}_1\\ \ddot{v}_2\\ \ddot{\theta}_2\end{Bmatrix} + [k^{(e)}]\begin{Bmatrix}v_1\\ \theta_1\\ v_2\\ \theta_2\end{Bmatrix} = -\int_0^L [N]^T q(x,t)\,\mathrm{d}x + \begin{Bmatrix}-V_1(t)\\ -M_1(t)\\ V_2(t)\\ M_2(t)\end{Bmatrix} \tag{10.79}$$

and all quantities are as previously defined. In the dynamic case, the nodal shear forces and bending moments may be time dependent, as indicated.

Assembly procedures for the beam element including the mass matrix are identical to those for the static equilibrium case. The global mass matrix is directly assembled, using the individual element mass matrices in conjunction with the element-to-global displacement relations. While system assembly is procedurally straightforward, the process is tedious when carried out by hand. Consequently, a complex example is not attempted. Instead, a relatively simple example of natural frequency determination is examined.

EXAMPLE 10.5

Using a single finite element, determine the natural circular frequencies of vibration of a cantilevered beam of length L, assuming constant values of ρ, E, and A.

■ Solution

The beam is depicted in Figure 10.10, with node 1 at the fixed support such that the boundary (constraint) conditions are $v_1 = \theta_1 = 0$. For free vibration, applied force and bending moment at the free end (node 2) are $V_2 = M_2 = 0$ and there is no applied distributed load. Under these conditions, the first two equations represented by Equation 10.79 are constraint equations and not of interest. Using the constraint conditions and the known applied forces, the last two equations are

$$\frac{\rho AL}{420}\begin{bmatrix}156 & -22L\\ -22L & 4L^2\end{bmatrix}\begin{Bmatrix}\ddot{v}_2\\ \ddot{\theta}_2\end{Bmatrix} + \frac{EI_z}{L^3}\begin{bmatrix}12 & -6L\\ -6L & 4L^2\end{bmatrix}\begin{Bmatrix}v_2\\ \theta_2\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$$

For computational convenience, the equations are rewritten as

$$\begin{bmatrix}156 & -22L\\ -22L & 4L^2\end{bmatrix}\begin{Bmatrix}\ddot{v}_2\\ \ddot{\theta}_2\end{Bmatrix} + \frac{420EI_z}{mL^3}\begin{bmatrix}12 & -6L\\ -6L & 4L^2\end{bmatrix}\begin{Bmatrix}v_2\\ \theta_2\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$$

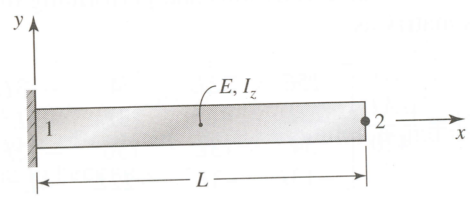

Figure 10.10 The cantilevered beam of Example 10.5 modeled as one element.

with $m = \rho AL$ representing the total mass of the beam. Assuming a sinusoidal displacement response, the frequency equation becomes

$$\begin{vmatrix} 12\lambda - 156\omega^2 & -6\lambda L + 22\omega^2 L \\ -6\lambda L + 22\omega^2 L & 4L^2(\lambda - \omega^2) \end{vmatrix} = 0$$

with $\lambda = 420EI_z/mL^3$. After expanding the determinant and performing considerable algebraic manipulation, the frequency equation becomes

$$5\omega^4 - 102\lambda\omega^2 + 3\lambda^2 = 0$$

Solving as a quadratic in ω^2, the roots are

$$\omega_1^2 = 0.02945\lambda \qquad \omega_2^2 = 20.37\lambda$$

Substituting for λ in terms of the beam physical parameters, we obtain

$$\omega_1 = 3.517\sqrt{\frac{EI_z}{mL^3}} \qquad \omega_2 = 92.50\sqrt{\frac{EI_z}{mL^3}} \text{ rad/sec}$$

as the finite element approximations to the first two natural circular frequencies. For comparison, the exact solution gives

$$\omega_1^{\text{exact}} = 3.516\sqrt{\frac{EI_z}{mL^3}} \qquad \omega_2^{\text{exact}} = 22.03\sqrt{\frac{EI_z}{mL^3}} \text{ rad/sec}$$

The fundamental frequency computed via a single element is essentially the same as the exact solution, whereas the second computed frequency is considerably larger than the corresponding exact value. As noted previously, a continuous system exhibits an infinite number of natural modes; we obtained only two modes in this example. If the number of elements is increased, the number of frequencies (natural modes) that can be computed increases as the number of degrees of freedom increases. In concert, the accuracy of the computed frequencies improves.

If the current example is refined by using two elements having length $L/2$ and the solution procedure repeated, we can compute four natural frequencies, the lowest two given by

$$\omega_1 = 3.516\sqrt{\frac{EI_z}{mL^3}} \qquad \omega_2 = 24.5\sqrt{\frac{EI_z}{mL^3}}$$

and we observe that the second natural circular frequency has improved (in terms of the exact solution) significantly. The third and fourth frequencies from this solution are found to be quite high in relation to the known exact values.

As indicated by the foregoing example, the number of natural frequencies and mode shapes that can be computed depend directly on the number of degrees of freedom of the finite element model. Also, as would be expected for convergence, as the number of degrees of freedom increases, the computed frequencies become closer to the exact values. As a general rule, the lower values (numerically) converge more rapidly to exact solution values. While this is discussed in more detail in conjunction with specific examples to follow, a general rule of thumb for frequency analysis is as follows: If the finite element analyst is interested in the first P modes of vibration of a structure, at least $2P$ modes should be calculated. Note that this implies the capability of calculating a subset of frequencies rather than all frequencies of a model. Indeed, this is possible and extremely important, since a practical finite element model may have thousands of degrees of freedom, hence thousands of natural frequencies. The computational burden of calculating all the frequencies is overwhelming and unnecessary, as is discussed further in the following section.

10.6 MASS MATRIX FOR A GENERAL ELEMENT: EQUATIONS OF MOTION

The previous examples dealt with relatively simple systems composed of linear springs and the bar and beam elements. In these cases, direct application of Newton's second law and Galerkin's finite element method led directly to the formulation of the matrix equations of motion; hence, the element mass matrices. For more general structural elements, an energy-based approach is preferred, as for static analyses. The approach to be taken here is based on *Lagrangian mechanics* and uses an energy method based loosely on *Lagrange's equations of motion* [4].

Prior to examining a general case, we consider the simple harmonic oscillator of Figure 10.1. At an arbitrary position x with the system assumed to be in motion, kinetic energy of the mass is

$$T = \frac{1}{2}m\dot{x}^2 \tag{10.80}$$

and the total potential energy is

$$U_e = \frac{1}{2}k(\delta_{st} + x)^2 - mg(\delta_{st} + x) \tag{10.81}$$

therefore, the total mechanical energy is

$$E_m = T + U_e = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k(\delta_{st} + x)^2 - mg(\delta_{st} + x) \tag{10.82}$$

As the simple harmonic oscillator model contains no mechanism for energy removal, the principle of conservation of mechanical energy applies; hence,

$$\frac{\mathrm{d}E_m}{\mathrm{d}t} = 0 = m\dot{x}\ddot{x} + k(\delta_{st} + x)\dot{x} - mg\dot{x} \tag{10.83}$$

or

$$m\ddot{x} + k(\delta_{st} + x) = mg \tag{10.84}$$

and the result is exactly the same as obtained via Newton's second law in Equation 10.2.