ES 128: Homework 2
Solutions

Problem 1

Show that the weak form of

$$\frac{d}{dx}(AE\frac{du}{dx}) + 2x = 0 \qquad \text{on } 1 < x < 3,$$
$$\sigma(1) = \left(E\frac{du}{dx}\right)_{x=1} = 0.1,$$
$$u(3) = 0.001$$

is given by

$$\int_1^3 \frac{dw}{dx} AE \frac{du}{dx} dx = -0.1(wA)_{x=1} + \int_1^3 2xw dx \qquad \forall w \text{ with } w(3) = 0.$$

Solution

We multiply the governing equation and the natural boundary condition over the domain [1, 3] by an arbitrary weight function:

$$\int_1^3 \left[w\left(\frac{d}{dx}\left(AE\frac{du}{dx} \right) + 2x \right) \right] dx = 0 \qquad \forall w(x), \tag{1.1}$$

$$\left(wA\left(E\frac{du}{dx} - 0.1 \right) \right)_{x=1} = 0 \qquad \forall w(1). \tag{1.2}$$

We integrate (1.1) by parts as

$$\int_1^3 \left[w\left(\frac{d}{dx}\left(AE\frac{du}{dx} \right) \right) \right] dx = \left(wAE\frac{du}{dx} \right)\Bigg|_{x=1}^{x=3} - \int_1^3 \left[\frac{dw}{dx} AE\frac{du}{dx} \right] dx. \tag{1.3}$$

Substituting (1.3) into (1.1) gives

$$-\int_1^3 \left[\frac{dw}{dx} AE\frac{du}{dx} \right] dx + \int_1^3 2wx dx + \left(wAE\frac{du}{dx} \right)\Bigg|_{x=3} - \left(wAE\frac{du}{dx} \right)\Bigg|_{x=1} = 0 \quad \forall w(x). \tag{1.4}$$

With $w(3) = 0$ and $\sigma(1) = 0.1$, we obtain

$$\int_1^3 \left[\frac{dw}{dx} AE\frac{du}{dx} \right] dx = \int_1^3 2wx dx - 0.1(wA)\big|_{x=1} \qquad \forall w(x) \text{ with } w(3) = 0. \tag{1.5}$$

Problem 2

Consider the (steel) bar in Figure 1. The bar has a uniform thickness t=1cm, Young's modulus E=200 $\times 10^9$ Pa, and weight density $\rho = 7\times 10^3\ kg/m^3$. In addition to its self-weight, the bar is subjected to a point load P=100N at its midpoint.

(a) Model the bar with two finite elements.
(b) Write down expressions for the element stiffness matrices and element body force vectors.
(c) Assemble the structural stiffness matrix K and global load vector F.
(d) Solve for the global displacement vector d.
(e) Evaluate the stresses in each element.
(f) Determine the reaction force at the support.

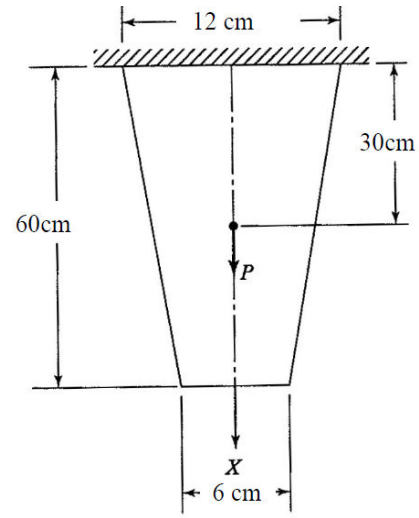

Figure 1

Solution

(a) Using two elements, each of 0.3m in length, we obtain the finite element model in Figure 1a. In this model, $x_1^{(1)} = 0$, $x_2^{(1)} = 0.3$, $x_1^{(2)} = 0.3$, $x_2^{(2)} = 0.6$, and $A(x)$=0.0012-0.001x.

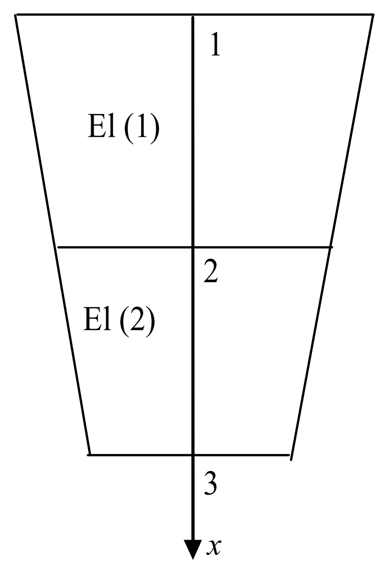

Figure 1a

(b) For element 1, $\mathrm{N}^{(1)} = \frac{1}{0.3}\begin{bmatrix} 0.3 - x & x \end{bmatrix}$,

$\mathrm{B}^{(1)} = \frac{1}{0.3}\begin{bmatrix} -1 & 1 \end{bmatrix}$, the element stiffness matrix is

$$\mathrm{K}^{(1)} = \int_0^{0.3} \mathrm{B}^{(1)\mathrm{T}} AE \mathrm{B}^{(1)} dx$$

$$= \frac{200\times 10^9}{0.09}\int_0^{0.3}(0.0012 - 0.001x)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx$$

$$= 10^9 \begin{bmatrix} 0.7 & -0.7 \\ -0.7 & 0.7 \end{bmatrix},$$

the element body force vector is

$$\mathrm{f}^{(1)} = \int_0^{0.3} \mathrm{N}^{(1)\mathrm{T}} \rho A dx + \left(\mathrm{N}^{(1)\mathrm{T}} P\right)\Big|_{x=0.3}$$

$$= \frac{7\times 10^3}{0.3}\int_0^{0.3}\begin{bmatrix} (0.3 - x)(0.0012 - 0.001x) \\ x(0.0012 - 0.001x) \end{bmatrix} dx + \begin{bmatrix} 0 \\ 100 \end{bmatrix}$$

$$= \begin{bmatrix} 1.155 \\ 101.05 \end{bmatrix}$$, and the scatter matrix is $\mathrm{L}^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

For element 2, $\mathrm{N}^{(2)} = \frac{1}{0.3}\begin{bmatrix}0.6-x & x-0.3\end{bmatrix}$, $\mathrm{B}^{(2)} = \frac{1}{0.3}\begin{bmatrix}-1 & 1\end{bmatrix}$, the element stiffness matrix is

$$\mathrm{K}^{(2)} = \int_{0.3}^{0.6} \mathrm{B}^{(2)\mathrm{T}} AE\mathrm{B}^{(2)} dx = \frac{200\times10^9}{0.09}\int_{0.3}^{0.6}(0.0012-0.001x)\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}dx$$

$= 10^9\begin{bmatrix}0.5 & -0.5\\ -0.5 & 0.5\end{bmatrix}$, the element body force vector

is $\mathrm{f}^{(2)} = \int_{0.3}^{0.6}\mathrm{N}^{(2)\mathrm{T}}\rho A dx = \frac{7\times10^3}{0.3}\int_{0.3}^{0.6}\begin{bmatrix}(0.6-x)(0.0012-0.001x)\\ (x-0.3)(0.0012-0.001x)\end{bmatrix}dx = \begin{bmatrix}0.84\\ 0.735\end{bmatrix}$, and

the scatter matrix is $\mathrm{L}^{(2)} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$.

(c) The global stiffness matrix is

$$\mathrm{K} = \sum_{e=1}^{2}\mathrm{L}^{e\mathrm{T}}\mathrm{K}^{e}\mathrm{L}^{e} = \mathrm{L}^{(1)\mathrm{T}}\mathrm{K}^{(1)}\mathrm{L}^{(1)} + \mathrm{L}^{(2)\mathrm{T}}\mathrm{K}^{(2)}\mathrm{L}^{(2)} = 10^9\begin{bmatrix}0.7 & -0.7 & 0\\ -0.7 & 1.2 & -0.5\\ 0 & -0.5 & 0.5\end{bmatrix}.$$

The global load vector is

$$\mathrm{f} = \sum_{e=1}^{2}\mathrm{L}^{e\mathrm{T}}\mathrm{f}^{e} = \mathrm{L}^{(1)\mathrm{T}}\mathrm{f}^{(1)} + \mathrm{L}^{(2)\mathrm{T}}\mathrm{f}^{(2)} = \begin{bmatrix}1.155\\ 101.89\\ 0.735\end{bmatrix}.$$

(d) Note that only the reaction force at node 1 is not zero, thus

$$\mathrm{f} + \mathrm{r} = \begin{bmatrix}r_1 + 1.155\\ 101.89\\ 0.735\end{bmatrix}.$$

The resulting global system of equations is

$$10^9\begin{bmatrix}0.7 & -0.7 & 0\\ -0.7 & 1.2 & -0.5\\ 0 & -0.5 & 0.5\end{bmatrix}\begin{bmatrix}0\\ u_2\\ u_3\end{bmatrix} = \begin{bmatrix}r_1 + 1.155\\ 101.89\\ 0.735\end{bmatrix}.$$

Solving the above equation,

$u_2 = 1.46607\times10^{-7}$ (m), $u_3 = 1.48077\times10^{-7}$ (m), and r_1 =-103.78(N).

(e) The stress field in element 1 is given by

$$\sigma^{(1)}(x) = E\mathrm{B}^{(1)}\mathrm{d}^{(1)} = 200\times10^9\times\frac{1}{0.3}[-1 \quad 1]\begin{bmatrix}0\\ 1.46607\times10^{-7}\end{bmatrix} = 9.7738\times10^4\,(\text{Pa}).$$

The stress field in element 2 is given by

$$\sigma^{(2)}(x) = E\mathrm{B}^{(2)}\mathrm{d}^{(2)} = 200\times10^9\times\frac{1}{0.3}[-1 \quad 1]\begin{bmatrix}1.46607\times10^{-7}\\ 1.48077\times10^{-7}\end{bmatrix} = 980\ (\text{Pa}).$$

(f) The reaction force at the support Node 1 is -103.78N.

Problem 3

Consider the mesh shown in Figure 2. The model consists of two linear displacement constant strain elements. The cross-sectional area is A=1, Young's modulus is E; both are constant. A body force $b(x)=cx$ is applied.

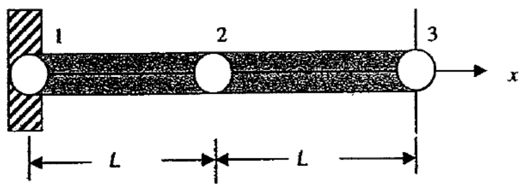

Figure 2

(a) Solve and plot $u(x)$ and $\varepsilon(x)$ for the FEM solution.

(b) Compare (by plotting) the finite element solution against the exact solution for the equation

$$E\frac{d^2u}{dx^2} = -b(x) = -cx.$$

(c) Solve the above problem using a single quadratic displacement element.

(d) Compare the accuracy of stress and displacement at the right end with that of two linear displacement elements.

(e) Check whether the equilibrium equation and traction boundary condition are satisfied for the two meshes.

Solution

(a) Using two linear displacement constant strain elements, each of l in length, we obtain the finite element model with $x_1^{(1)} = 0$, $x_2^{(1)} = l$, $x_1^{(2)} = l$, and $x_2^{(2)} = 2l$.

For element 1, $\mathrm{N}^{(1)} = \frac{1}{l}\begin{bmatrix} l-x & x \end{bmatrix}$, $\mathrm{B}^{(1)} = \frac{1}{l}\begin{bmatrix} -1 & 1 \end{bmatrix}$, the element stiffness matrix is $\mathrm{K}^{(1)} = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, the element body force vector is $\mathrm{f}^{(1)} = \int_0^l \mathrm{N}^{(1)\mathrm{T}} cx dx = \begin{bmatrix} cl^2/6 \\ cl^2/3 \end{bmatrix}$, and the scatter matrix is $\mathrm{L}^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

For element 2, $\mathrm{N}^{(2)} = \frac{1}{l}\begin{bmatrix} 2l-x & x-l \end{bmatrix}$, $\mathrm{B}^{(2)} = \frac{1}{l}\begin{bmatrix} -1 & 1 \end{bmatrix}$, the element stiffness matrix is $\mathrm{K}^{(2)} = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, the element body force vector is $\mathrm{f}^{(2)} = \int_l^{2l} \mathrm{N}^{(2)\mathrm{T}} cx dx = \begin{bmatrix} 2cl^2/3 \\ 5cl^2/6 \end{bmatrix}$, and the scatter matrix is $\mathrm{L}^{(2)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The global stiffness matrix is

$$\mathrm{K} = \sum_{e=1}^{2} \mathrm{L}^{e\mathrm{T}}\mathrm{K}^{e}\mathrm{L}^{e} = \mathrm{L}^{(1)\mathrm{T}}\mathrm{K}^{(1)}\mathrm{L}^{(1)} + \mathrm{L}^{(2)\mathrm{T}}\mathrm{K}^{(2)}\mathrm{L}^{(2)} = \frac{EA}{l}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

The global load vector is

$$\mathrm{f} = \sum_{e=1}^{2} \mathrm{L}^{eT}\mathrm{f}^{e} = \mathrm{L}^{(1)T}\mathrm{f}^{(1)} + \mathrm{L}^{(2)T}\mathrm{f}^{(2)} = cl^2 \begin{bmatrix} 0.1667 \\ 1.0 \\ 0.8333 \end{bmatrix}.$$

The resulting global system of equations is

$$\frac{EA}{l}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} r_1 + 0.1667cl^2 \\ cl^2 \\ 0.8333cl^2 \end{bmatrix}.$$

Solving the above equation,

$u_2 = 1.8333\dfrac{cl^3}{EA}$, $u_3 = 2.6666\dfrac{cl^3}{EA}$, and $r_1 = -2\,cl^2$.

When $0 \le x \le l$

$$u(x) = \mathrm{N}^{(1)}\mathrm{d}^{(1)} = \frac{1}{l}\begin{bmatrix} l-x & x \end{bmatrix}\begin{bmatrix} 0 \\ 1.8333\dfrac{cl^3}{EA} \end{bmatrix} = 1.8333\frac{cl^2 x}{EA},$$

and $\varepsilon(x) = \dfrac{du}{dx} = 1.8333\dfrac{cl^2}{EA}$.

When $l \le x \le 2l$

$$u(x) = \mathrm{N}^{(2)}\mathrm{d}^{(2)} = \frac{1}{l}\begin{bmatrix} 2l-x & x-l \end{bmatrix}\begin{bmatrix} 1.8333\dfrac{cl^3}{EA} \\ 2.6666\dfrac{cl^3}{EA} \end{bmatrix} = \frac{cl^3}{EA} + 0.8333\frac{cl^2}{EA}x,$$

and $\varepsilon(x) = \dfrac{du}{dx} = 0.8333\dfrac{cl^2}{EA}$.

(b). The governing equation is

$$EA\frac{d^2u}{dx^2} = -b(x) = -cx,$$

where A=1. The boundary condition is $u(0)$=0, and $\sigma(2l) = 0$.

Solving this linear ODE, we obtain the exact solution $u(x) = -\dfrac{c}{6EA}x^3 + c_1 x + c_2$.

Since u(0)=0, $c_2 = 0$. Since $\sigma(2l) = 0$, $c_1 = \dfrac{2cl^2}{EA}$. Thus

$$u(x) = -\frac{c}{6EA}x^3 + \frac{2cl^2}{EA}x.$$

$$\varepsilon(x) = -\frac{cx^2}{2EA} + \frac{2cl^2}{EA}$$

The comparisons between the approximation results and the exact solutions are shown in Figures 2a (displacement), and 2b (strain).

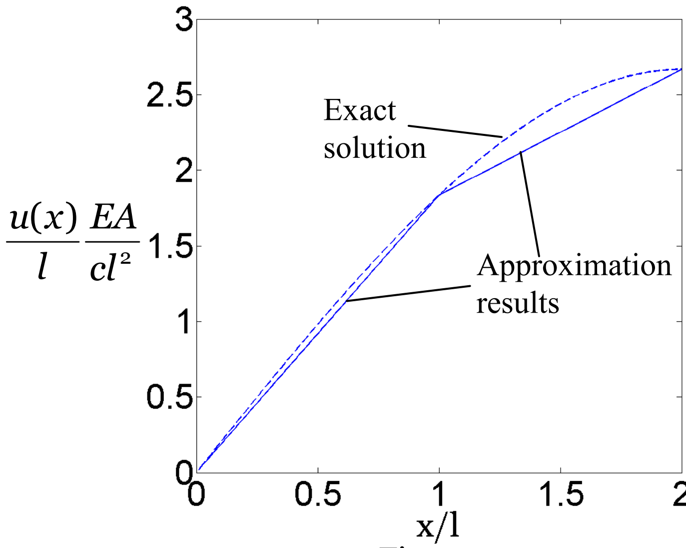

Figure 2a

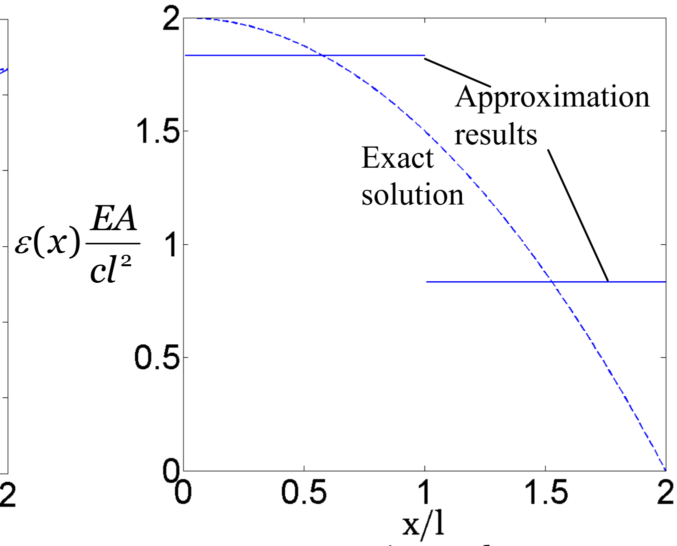

Figure 2b

(c) Using a single quadratic displacement element with $x_1^{(1)} = 0$, $x_2^{(1)} = l$, and $x_3^{(1)} = 2l$, the element shape functions are

$$\mathrm{N}_1^{(1)} = \frac{\left(x - x_2^{(1)}\right)\left(x - x_3^{(1)}\right)}{\left(x_1^{(1)} - x_2^{(1)}\right)\left(x_1^{(1)} - x_3^{(1)}\right)} = \frac{(x-l)(x-2l)}{(-l)(-2l)} = \frac{(x-l)(x-2l)}{2l^2},$$

$$\mathrm{N}_2^{(1)} = \frac{\left(x - x_1^{(1)}\right)\left(x - x_3^{(1)}\right)}{\left(x_2^{(1)} - x_1^{(1)}\right)\left(x_2^{(1)} - x_3^{(1)}\right)} = \frac{x(x-2l)}{l(-l)} = \frac{x(x-2l)}{-l^2},$$

$$\mathrm{N}_3^{(1)} = \frac{\left(x - x_1^{(1)}\right)\left(x - x_2^{(1)}\right)}{\left(x_3^{(1)} - x_1^{(1)}\right)\left(x_3^{(1)} - x_2^{(1)}\right)} = \frac{x(x-l)}{2l^2} = \frac{x(x-l)}{2l^2}.$$

The corresponding B-matrix is

$$\mathrm{B}_1^{(1)} = \frac{d\mathrm{N}_1^{(1)}}{dx} = \frac{x - 3l/2}{l^2},$$

$$\mathrm{B}_2^{(1)} = \frac{d\mathrm{N}_2^{(1)}}{dx} = \frac{2x - 2l}{-l^2},$$

$$\mathrm{B}_3^{(1)} = \frac{d\mathrm{N}_3^{(1)}}{dx} = \frac{x - l/2}{l^2}.$$

The element stiffness matrix is

$$\mathrm{K}^{(1)} = \int_0^{2l} \mathrm{B}^{(1)\mathrm{T}} EA \mathrm{B}^{(1)} dx = EA \int_2^{2l} \begin{bmatrix} \frac{x-3l/2}{l^2} \\ \frac{2x-2l}{-l^2} \\ \frac{x-l/2}{l^2} \end{bmatrix} \begin{bmatrix} \frac{x-3l/2}{l^2} & \frac{2x-2l}{-l^2} & \frac{x-l/2}{l^2} \end{bmatrix} dx$$

$$= EA\int_0^{2l}\begin{bmatrix} \left(\frac{x-3l/2}{l^2}\right)^2 & \left(\frac{x-3l/2}{l^2}\right)\left(\frac{2x-2l}{-l^2}\right) & \left(\frac{x-3l/2}{l^2}\right)\left(\frac{x-l/2}{l^2}\right) \\ \left(\frac{x-3l/2}{l^2}\right)\left(\frac{2x-2l}{-l^2}\right) & \left(\frac{2x-2l}{-l^2}\right)^2 & \left(\frac{2x-2l}{-l^2}\right)\left(\frac{x-l/2}{l^2}\right) \\ \left(\frac{x-3l/2}{l^2}\right)\left(\frac{x-l/2}{l^2}\right) & \left(\frac{2x-2l}{-l^2}\right)\left(\frac{x-l/2}{l^2}\right) & \frac{x-l/2}{l^2} \end{bmatrix} dx$$

$$= \frac{EA}{l}\begin{bmatrix} 7/6 & -4/3 & 1/6 \\ -4/3 & 8/3 & -4/3 \\ 1/6 & -4/3 & 7/6 \end{bmatrix}.$$

The element body force vector is $\mathrm{f}^{(1)} = \int_0^{2l} \mathrm{N}^{(1)\mathrm{T}} cxdx = \begin{bmatrix} 0 \\ 4cl^2/3 \\ 2cl^2/3 \end{bmatrix}$.

The resulting global system of equations is

$$\frac{EA}{l}\begin{bmatrix} 7/6 & -4/3 & 1/6 \\ -4/3 & 8/3 & -4/3 \\ 1/6 & -4/3 & 7/6 \end{bmatrix}\begin{bmatrix} 0 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} r_1 \\ 4cl^2/3 \\ 2cl^2/3 \end{bmatrix}.$$

Solving the above equation, we obtain

$u_2 = 1.8333\frac{cl^3}{EA}$, $u_3 = 2.6666\frac{cl^3}{EA}$, and r_1=-2cl^2.

$$u(x) = \mathrm{N}^{(1)}\mathrm{d}^{(1)} = \begin{bmatrix} \frac{(x-l)(x-2l)}{2l^2} & \frac{x(x-2l)}{-l^2} & \frac{x(x-l)}{2l^2} \end{bmatrix}\begin{bmatrix} 0 \\ 1.8333\frac{cl^3}{EA} \\ 2.6666\frac{cl^3}{EA} \end{bmatrix},$$

$$= -\frac{0.5cl}{EA}x^2 + 2.3333\frac{cl^2}{EA}x.$$

$$\varepsilon(x) = -\frac{cl}{EA}x + 2.3333\frac{cl^2}{EA}.$$

(d) At the right end with x=2l, for both of two linear displacement elements and a single quadratic displacement element, the displacements are same to the exact result ($u_3 = 2.6666\frac{cl^3}{EA}$). As for the stress and strain, with two linear displacement elements, we obtain $\varepsilon(2l) = 0.8333\frac{cl^2}{EA}$ and $\sigma(2l) = 0.8333\frac{cl^2}{A}$.

With a single quadratic displacement element, $\varepsilon(2l) = 0.3333\frac{cl^2}{EA}$, and $\sigma(2l) = 0.3333\frac{cl^2}{A}$. It is found that the approximation result of the stress and strain with a single quadratic displacement element is closer to the exact result ($\sigma = 0$ and $\varepsilon = 0$).

(e). For both of two linear displacement elements and a single quadratic displacement element, the reaction forces acting on node 1 are r_1 =-2 cl^2, which satisfy the equilibrium equation ($r_1 + \int_0^{2l} cxdx = 0$). However, as shown in (d), the stress boundary conditions are not satisfied for the two meshes.