ES128: Homework 3
Solutions

Problem 1

Derive the equilibrium equations and the relations between strains ($\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{zz}, \varepsilon_{r\theta}, \varepsilon_{rz}, \varepsilon_{\theta z}$) and displacements in cylindrical coordinates with $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, and $x_3 = z$.

Solution

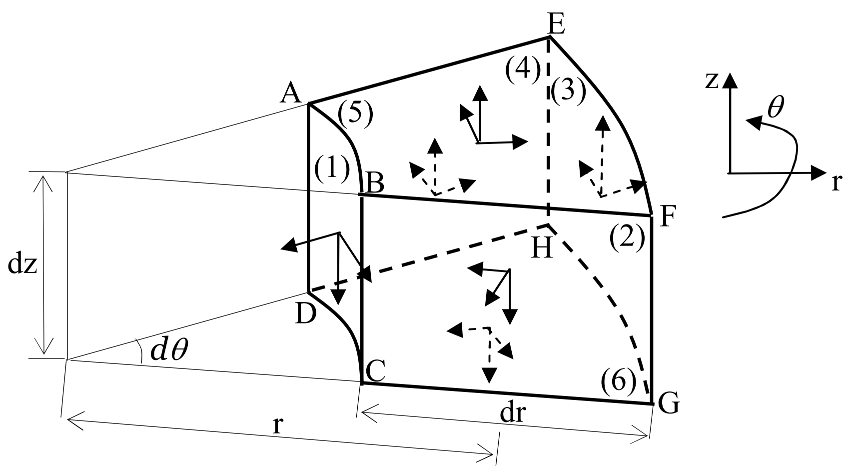

Fig 1

Fig. 1 shows an element ABCDEFGH of with 6 surfaces. In surface (1) (surface ABCD), the stresses in r, θ, and z directions are

$$\sigma_r^{(1)} = \sigma_r - \frac{\partial \sigma_r}{\partial r}\frac{dr}{2},$$

$$\sigma_{r\theta}^{(1)} = \sigma_{r\theta} - \frac{\partial \sigma_{r\theta}}{\partial r}\frac{dr}{2},$$

$$\sigma_{rz}^{(1)} = \sigma_{rz} - \frac{\partial \sigma_{rz}}{\partial r}\frac{dr}{2}.$$

On surface (2) (surface BCFG), the stresses in θ, r, and z directions are

$$\sigma_\theta^{(2)} = \sigma_\theta - \frac{\partial \sigma_\theta}{\partial \theta}\frac{d\theta}{2},$$

$$\sigma_{\theta r}^{(2)} = \sigma_{\theta r} - \frac{\partial \sigma_{\theta r}}{\partial \theta}\frac{d\theta}{2},$$

$$\sigma_{\theta z}^{(2)} = \sigma_{\theta z} - \frac{\partial \sigma_{\theta z}}{\partial \theta}\frac{d\theta}{2}.$$

On surface (3) (surface EFGH), the stresses in r, θ, and z directions are

$$\sigma_r^{(3)} = \sigma_r + \frac{\partial \sigma_r}{\partial r}\frac{dr}{2},$$
$$\sigma_{r\theta}^{(3)} = \sigma_{r\theta} + \frac{\partial \sigma_{r\theta}}{\partial r}\frac{dr}{2},$$
$$\sigma_{rz}^{(3)} = \sigma_{rz} + \frac{\partial \sigma_{rz}}{\partial r}\frac{dr}{2}.$$
On surface (4) (surface AEHD), the stresses in θ, r, and z directions are
$$\sigma_\theta^{(4)} = \sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta}\frac{d\theta}{2},$$
$$\sigma_{\theta r}^{(4)} = \sigma_{\theta r} + \frac{\partial \sigma_{\theta r}}{\partial \theta}\frac{d\theta}{2},$$
$$\sigma_{\theta z}^{(4)} = \sigma_{\theta z} + \frac{\partial \sigma_{\theta z}}{\partial \theta}\frac{d\theta}{2}.$$
On surface (5) (surface AEFB), the stresses in z, r, and θ directions are
$$\sigma_z^{(5)} = \sigma_z + \frac{\partial \sigma_z}{\partial z}\frac{dz}{2},$$
$$\sigma_{zr}^{(5)} = \sigma_{zr} + \frac{\partial \sigma_{zr}}{\partial z}\frac{dz}{2},$$
$$\sigma_{z\theta}^{(5)} = \sigma_{z\theta} + \frac{\partial \sigma_{zr}}{\partial z}\frac{dz}{2}.$$
On surface (6) (surface DHGC), the stresses in z, r, and θ directions are
$$\sigma_z^{(6)} = \sigma_z - \frac{\partial \sigma_z}{\partial z}\frac{dz}{2},$$
$$\sigma_{zr}^{(6)} = \sigma_{zr} - \frac{\partial \sigma_{zr}}{\partial z}\frac{dz}{2},$$
$$\sigma_{z\theta}^{(6)} = \sigma_{z\theta} - \frac{\partial \sigma_{zr}}{\partial z}\frac{dz}{2}.$$
The conditions that the momentums are zero yield that
$\sigma_{r\theta} = \sigma_{\theta r}$; $\sigma_{\theta z} = \sigma_{z\theta}$; $\sigma_{ra} = \sigma_{zr}$.
In the r direction, the resultant force is zero. Thus

$$\left(\sigma_r + \frac{\partial \sigma_r}{\partial r}\frac{dr}{2}\right)dz\left(r + \frac{dr}{2}\right)d\theta - \left(\sigma_r - \frac{\partial \sigma_r}{\partial r}\frac{dr}{2}\right)dz\left(r - \frac{dr}{2}\right)d\theta$$
$$+\left(\sigma_{\theta r} + \frac{\partial \sigma_{\theta r}}{\partial \theta}\frac{d\theta}{2}\right)dzdr\cos\left(\frac{\theta}{2}\right) - \left(\sigma_{\theta r} - \frac{\partial \sigma_{\theta r}}{\partial \theta}\frac{d\theta}{2}\right)dzdr\cos\left(\frac{\theta}{2}\right)$$
$$-\left(\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta}\frac{d\theta}{2}\right)dzdr\sin\left(\frac{\theta}{2}\right) - \left(\sigma_\theta - \frac{\partial \sigma_\theta}{\partial \theta}\frac{d\theta}{2}\right)dzdr\sin\left(\frac{\theta}{2}\right)$$
$$+\left(\sigma_{zr} + \frac{\partial \sigma_{zr}}{\partial z}\frac{dz}{2}\right)rd\theta dr - \left(\sigma_{zr} - \frac{\partial \sigma_{zr}}{\partial z}\frac{dz}{2}\right)rd\theta dr = 0.$$
Since $\sin(\theta/2) \approx \theta/2$ and $\cos(\theta/2) \approx 1$, we obtain

$$\frac{\partial \sigma_r}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta}+\frac{\partial \sigma_{zr}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}=0\,.$$

Similarly, the condition that the resultant force is zero in the θ direction gives

$$\frac{\partial \sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta}}{\partial \theta}+\frac{\partial \sigma_{z\theta}}{\partial z}+\frac{2\sigma_{r\theta}}{r}=0\,.$$

The condition that the resultant force is zero in the z direction gives

$$\frac{\partial \sigma_{rz}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta}+\frac{\partial \sigma_{z}}{\partial z}+\frac{\sigma_{rz}}{r}=0\,.$$

Next, let us analyze the relations between strains and displacements in cylindrical coordinates

Consider first the displacement in the r-direction, u_r. We see from Fig. 2a that

$$e_{rr}=\frac{u_r+(\partial u_r/\partial r)dr-u_r}{dr}=\frac{\partial u_r}{\partial r}.$$

From the same figure, we see also that a radial displacement of a circumferential element causes an elongation of that element and, hence, a strain in the θ-direction. The element ab, which was originally of length $r\,d\theta$, is displaced to $a'b'$ and becomes of length $(r+u_r)\,d\theta$. The tangential strain due to this radial displacement is, therefore,

$$e_{\theta\theta}^{(1)}=\frac{(r+u_r)\,d\theta-r\,d\theta}{r\,d\theta}=\frac{u_r}{r}.$$

On the other hand, as shown in Fig. 2b , the tangential displacement u_θ gives rise to a tangential strain equal to

$$e_{\theta\theta}^{(2)}=\frac{u_\theta+(\partial u_\theta/\partial\theta)\,d\theta-u_\theta}{r\,d\theta}=\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}.$$

The total tangential strain is

$$e_{\theta\theta}=\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}.$$

The normal strain in the axial direction is

$$e_{zz}=\frac{\partial u_z}{\partial z},$$

as in the case of rectangular coordinates.

The shearing strain $e_{r\theta}$ is equal to one-half of the change of angle $\angle C'a'b' - \angle Cab$, as illustrated in Fig. 2c . A direct examination of the figure shows that

$$e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right).$$

The first term comes from the change in the radial displacement in the θ-direction, the second term comes from the change in the tangential displacement in the radial direction, and the last term appears since part of the change in slope of the line $a'C'$ comes from the rotation of the element as a solid body about the axis through O.

The remaining strain components, $e_{z\theta}$ and e_{zr} can be derived with reference to Figs. 2b and 2e. We have

$$e_{z\theta} = \frac{1}{2}\left[\frac{(\partial u_z/\partial\theta)d\theta}{r\,d\theta} + \frac{(\partial u_\theta/\partial z)dz}{dz}\right] = \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_z}{\partial\theta} + \frac{\partial u_\theta}{\partial z}\right]$$

and

$$e_{zr} = \frac{1}{2}\left[\frac{(\partial u_r/\partial z)dz}{dz} + \frac{(\partial u_z/\partial r)dr}{dr}\right] = \frac{1}{2}\left[\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right].$$

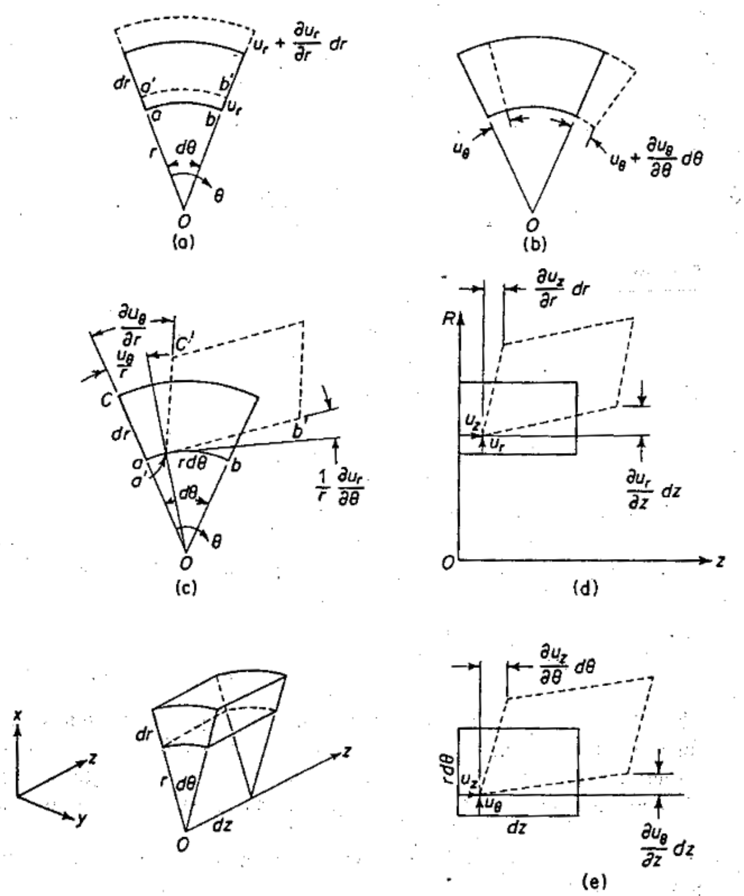

Fig. 2 Displacement in cylindrical polar coordinates. (From E. E. Sechler, *Elasticity in Engineering*, Courtesy Mrs. Magaret Sechler.) A free-body diagram of an infinitesimal element of material and two systems of coordinates are shown at the lower left corner. (a) Radial strain due to variation of the radial displacement field in the radial direction. (b) Circumferential strain due to variation of circumferential displacement in the circumferential direction. (c) $\partial u_\theta/\partial r$ and $(1/r)\partial u_r/\partial\theta$ cause shear strain $e_{r\theta}$. (d) $\partial u_z/\partial r$ and $\partial u_r/\partial z$ cause shear strain e_{rz}. (e) $(1/r)\partial u_z/\partial\theta$ and $\partial u_\theta/\partial z$ cause shear strain $e_{z\theta}$.

Problem 2

Elastic bulk modulus B is defined as the stiffness under hydrostatic pressure, namely,

$$p = -B\frac{\Delta V}{V}$$

Here $\Delta V / V$ is the volumetric strain caused by the hydrostatic pressure p.

(a) Show that the volumetric strain $\Delta V / V$ relates to the axial strains as

$$\frac{\Delta V}{V} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}.$$

(b) Express the bulk modulus B in terms of Young's modulus and Poisson's ratio.

(c) What is the value of Poisson's ratio when the material is incompressible? Interpret your result under uniaxial stress state.

(d) Show that Poisson's ratio must be smaller than ½. What would happen if Poisson's ratio were greater than ½?

Solution

(a) Assume that the original dimensions are X Y Z in x, y, and z directions, respectively. After deformation, the current dimensions are X+ΔX, Y+ΔY, and Z+ΔZ, respectively. The volumetric strain

$$\frac{\Delta V}{V} = \frac{(X+\Delta X)(Y+\Delta Y)(Z+\Delta Z) - XYZ}{XYZ},$$

where ΔX, ΔY, ΔZ are assumed to be small, compared to X, Y, Z, respectively. We only keep the linear terms in the above equation

$$\frac{\Delta V}{V} = \frac{XY\Delta Z + XZ\Delta Y + YZ\Delta X}{XYZ} = \frac{\Delta X}{X} + \frac{\Delta Y}{Y} + \frac{\Delta Z}{Z} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

(b) With Hooke's law $\varepsilon_x = \frac{1}{E}\left(\sigma_x - \upsilon\left(\sigma_y + \sigma_z\right)\right)$, $\varepsilon_y = \frac{1}{E}\left(\sigma_y - \upsilon\left(\sigma_x + \sigma_z\right)\right)$, and $\varepsilon_z = \frac{1}{E}\left(\sigma_z - \upsilon\left(\sigma_x + \sigma_y\right)\right)$, we obtain that

$$\varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{1-2\upsilon}{E}(\sigma_x + \sigma_y + \sigma_z),$$

Under hydrostatic pressure, $\sigma_x = \sigma_y = \sigma_z = -p$, thus,

$$\frac{\Delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{1-2\upsilon}{E}(-3p) = -p / B$$

$$B = \frac{E}{3(1-2\upsilon)}$$

(c) When υ=0.5, the material is incompressible. For uniaxial stress state, $\varepsilon_x = \frac{1}{E}\sigma_x$; $\varepsilon_y = \varepsilon_z = -\frac{\upsilon}{E}\sigma_x = -\frac{1}{2E}\sigma_x$. Thus, $\frac{\Delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z = 0$.

(d) Since $\frac{\Delta V}{V} = \frac{1-2\upsilon}{E}(-3p)$. Thus υ must be smaller than 1/2. If υ is greater than 1/2, with hydrostatic pressure, the volume of the body can increase.

Problem 3

Stress concentration at geometric discontinuities is the most important practical result in elasticity theory. For example, for a small circular hole in a large plate under uniaxial stress *S*, the elasticity solution gives the hoop stress around the hole:

$$\sigma_{\theta\theta} = S(1 - 2\cos 2\theta)$$

Here the polar angle θ is measured from the loading direction. The problem is solved in many elasticity textbooks.

a) Under uniaxial tension, indicate the highest tensile stress around the hole.
b) Under uniaxial compression, indicate the highest tensile stress around the hole.
c) Calculate the stress concentration at the hole when the plate is under a pure shear stress. Use the above solution and linear superposition.

Solution

(a) Under uniaxial tension, the tensile stress is the highest at $\theta = \pi/2$ ($\sigma_{\theta\theta} = 3S$).

(b) Under uniaxial compression, the tensile stress is the highest at $\theta = 0$ ($\sigma_{\theta\theta} = -S$).

(c)

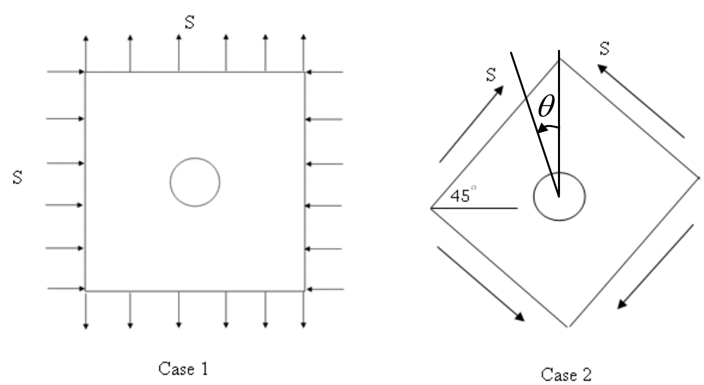

Fig. 3

In Fig. 3, case 1 is exactly same as case 2 (pure shear). The reason is as follows. Take a triangle free body from case 1 as shown if Fig.4.

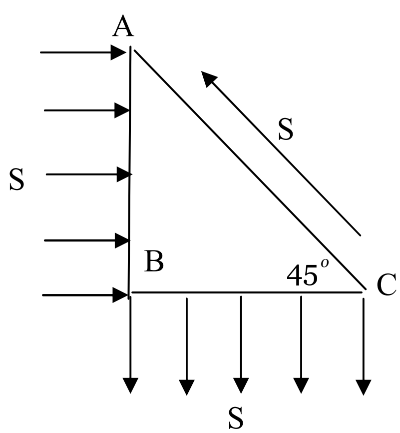

Fig. 4

Based on the force balance, it is easy to show that only the shear stress S is applied on the surface AC, and the normal stress is zero. Thus case 2 can be reduced to an easier problem of case 1. Based on linear superposition,

$\sigma_{\theta\theta} = S(1-2\cos 2\theta) - S(1-2\cos 2(\theta+\pi/2)) = -4S\cos 2\theta$,

where θ is shown in Fig.3. The tensile stress is the highest at $\theta = \pi/2$ ($\sigma_{\theta\theta} = 4S$).

Problem 4

An elastic layer is sandwiched between two perfectly rigid plates, to which it is bonded. The layer is compressed between the plates, the direct stress being σ_z. Supposing that the attachment to the plates prevents lateral strain ε_x, ε_y completely, find the apparent Young's modulus (that is σ_z/ε_z) in terns of *E* and *υ*. Show that it is many times *E* if the material of the layer has a Poisson's ratio only slightly less than 0.5, e.g., rubber.

Solution

The shear stresses vanish, but all the three axial stresses σ_x, σ_y, σ_z are nonzero. By symmetry, we note that

$$\sigma_x = \sigma_y$$

Because the elastic layer is bonded to the rigid plate, the two components of strain vanish:

$$\varepsilon_x = \varepsilon_y = 0$$

That is, the elastic layer is in a state of uniaxial strain: $\varepsilon_z \neq 0$. Using Hooke's law, we obtain that

$$0 = \varepsilon_x = \frac{1}{E}\left(\sigma_x - \upsilon\sigma_y - \upsilon\sigma_z\right)$$

or

$$\sigma_x = \frac{\upsilon}{1-\upsilon}\sigma_z$$

Using Hooke's law again, we obtain that

$$\varepsilon_z = \frac{1}{E}\left(\sigma_z - \upsilon\left(\sigma_x + \sigma_y\right)\right) = \frac{(1+\upsilon)(1-2\upsilon)}{(1-\upsilon)E}\sigma_z$$

The apparent Young's modulus

$$\frac{\sigma_z}{\varepsilon_z} = \frac{(1-\upsilon)E}{(1+\upsilon)(1-2\upsilon)}.$$

When *υ* is only slightly less than 0.5, e.g., for rubbers, the apparent Young's modulus is many times *E*.