ES128: Homework 4
Solutions

Problem 1

Consider a one-element triangular mesh shown in Fig. 1. The boundary conditions are as follows. The edge BC is constrained in y and traction free in x, whereas the edge AB is constrained in x and traction free in y. The edge AC is subject to traction normal to the edge as shown in Fig. 1. Assume Young's modulus E= 3×10^{11} Pa and Poisson's ratio v=0.3.

a. Construct the stiffness matrix.
b. Calculate the global force matrix.
c. Solve for the unknown displacement matrix and calculate the stress at (1.5, 1.5).

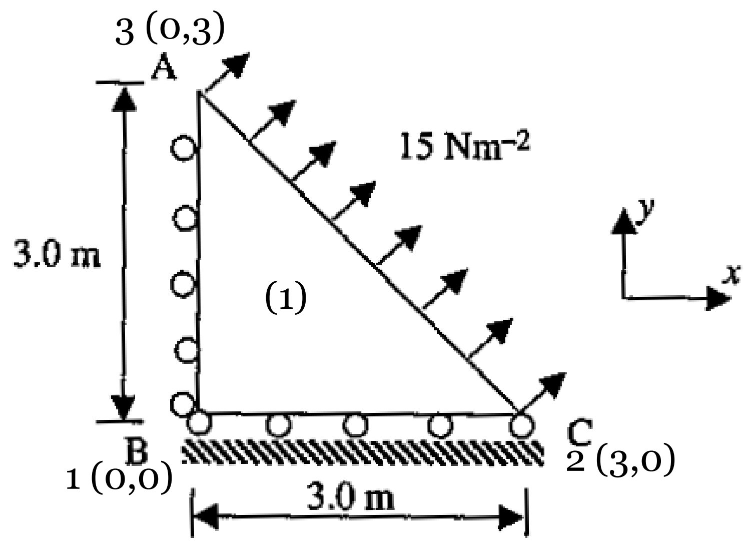

Fig. 1 Triangular domain with mixed boundary conditions

Solution

We will use the plane strain and plane stress conditions to solve this problem. As shown in Fig.1, $x_1^{(1)}=0$, $y_1^{(1)}=0$, $x_2^{(1)}=3$, $y_2^{(1)}=0$, $x_3^{(1)}=0$, and $y_3^{(1)}=3$.

First, with the plane strain condition

$$D^{(1)}=\frac{E}{(1+v)(1-2v)}\begin{bmatrix}1-v & v & 0\\ v & 1-v & 0\\ 0 & 0 & (1-2v)/2\end{bmatrix}=5.77\times10^{11}\begin{bmatrix}0.7 & 0.3 & 0\\ 0.3 & 0.7 & 0\\ 0 & 0 & 0.2\end{bmatrix}.$$

The shape functions are

$$N_1^{(1)} = \frac{1}{2A^{(1)}}\left(x_2^{(1)}y_3^{(1)} - x_3^{(1)}y_2^{(1)} + \left(y_2^{(1)} - y_3^{(1)}\right)x + \left(x_3^{(1)} - x_2^{(1)}\right)y\right) = 1 - \frac{1}{3}x - \frac{1}{3}y\,,$$

$$N_2^{(1)} = \frac{1}{2A^{(1)}}\left(x_3^{(1)}y_1^{(1)} - x_1^{(1)}y_3^{(1)} + \left(y_3^{(1)} - y_1^{(1)}\right)x + \left(x_1^{(1)} - x_3^{(1)}\right)y\right) = \frac{1}{3}x\,,$$

$$N_3^{(1)} = \frac{1}{2A^{(1)}}\left(x_1^{(1)}y_2^{(1)} - x_2^{(1)}y_1^{(1)} + \left(y_1^{(1)} - y_2^{(1)}\right)x + \left(x_2^{(1)} - x_1^{(1)}\right)y\right) = \frac{1}{3}y\,.$$

The strain-displacement matrix is

$$\mathrm{B}^{(1)} = \begin{bmatrix} \frac{\partial N_1^{(1)}}{\partial x} & 0 & \frac{\partial N_2^{(1)}}{\partial x} & 0 & \frac{\partial N_3^{(1)}}{\partial x} & 0 \\ 0 & \frac{\partial N_1^{(1)}}{\partial y} & 0 & \frac{\partial N_2^{(1)}}{\partial y} & 0 & \frac{\partial N_3^{(1)}}{\partial y} \\ \frac{\partial N_1^{(1)}}{\partial y} & \frac{\partial N_1^{(1)}}{\partial x} & \frac{\partial N_2^{(1)}}{\partial y} & \frac{\partial N_2^{(1)}}{\partial x} & \frac{\partial N_3^{(1)}}{\partial y} & \frac{\partial N_3^{(1)}}{\partial x} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}.$$

The stiff matrix is

$$\mathrm{K}=\mathrm{K}^{(1)} = \int_\Omega \mathrm{B}^{(1)\mathrm{T}}\mathrm{D}^{(1)}\mathrm{B}^{(1)}d\Omega = A^{(1)}\mathrm{B}^{(1)\mathrm{T}}\mathrm{D}^{(1)}\mathrm{B}^{(1)}$$

$$= 10^{11}\begin{array}{c} \begin{array}{cccccc} \mathrm{1x} & \mathrm{1y} & \mathrm{2x} & \mathrm{2y} & \mathrm{3x} & \mathrm{3y} \end{array} \\ \begin{bmatrix} 2.5965 & 1.4425 & -2.0195 & -0.5770 & -0.5770 & -0.8655 \\ 1.4425 & 2.5965 & -0.8655 & -0.5770 & -0.5770 & -2.0195 \\ -2.0195 & -0.8655 & 2.0195 & 0 & 0 & 0.8655 \\ -0.5770 & -0.5770 & 0 & 0.5770 & 0.5770 & 0 \\ -0.5770 & -0.5770 & 0 & 0.5770 & 0.5770 & 0 \\ -0.8655 & -2.0195 & 0.8655 & 0 & 0 & 2.0195 \end{bmatrix} \end{array} \begin{array}{c} \\ 1x \\ 1y \\ 2x \\ 2y \\ 3x \\ 3y \end{array}.$$

The force vector is

$$\mathrm{f}_\Gamma^{(1)} = \int_{\Gamma_{23}} N^{(1)T}\bar{t}\,d\Gamma = \int_{\Gamma_{23}} \begin{bmatrix} 1 - \frac{1}{3}x - \frac{1}{3}y & 0 \\ 0 & 1 - \frac{1}{3}x - \frac{1}{3}y \\ \frac{1}{3}x & 0 \\ 0 & \frac{1}{3}x \\ \frac{1}{3}y & 0 \\ 0 & \frac{1}{3}y \end{bmatrix} \begin{bmatrix} 7.5\sqrt{2} \\ 7.5\sqrt{2} \end{bmatrix} d\Gamma\,.$$

At the edge CA, x=-y+3, $d\Gamma=\sqrt{2}dy$ Substituting x=-y+3 and $d\Gamma=\sqrt{2}dy$ into the above equation yields

$$\mathrm{f}_\Gamma^{(1)}=\int_0^3\begin{bmatrix}0 & 0\\0 & 0\\-\frac{1}{3}y+1 & 0\\0 & -\frac{1}{3}y+1\\\frac{1}{3}y & 0\\0 & \frac{1}{3}y\end{bmatrix}dy\begin{bmatrix}15\\15\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & 0\\1.5 & 0\\0 & 1.5\\1.5 & 0\\0 & 1.5\end{bmatrix}\begin{bmatrix}15\\15\end{bmatrix}=\begin{array}{c}\begin{bmatrix}0\\0\\22.5\\22.5\\22.5\\22.5\end{bmatrix}\end{array}\begin{array}{c}1x\\1y\\2x\\2y\\3x\\3y\end{array}.$$

In addition, $\mathrm{d}^{(1)}=\begin{bmatrix}0\\0\\u_{2x}\\0\\0\\u_{3y}\end{bmatrix}$; $\mathrm{f}_\Gamma^{(1)}+\mathrm{r}^{(1)}=\begin{bmatrix}r_{1x}\\r_{1y}\\22.5\\r_{2y}+22.5\\r_{3x}+22.5\\22.5\end{bmatrix}$.

The global stiff matrix is

$$\begin{array}{cccccc}1x & 1y & 2y & 3x & 2x & 3y\end{array}$$

$$10^{11}\begin{bmatrix}2.5965 & 1.4425 & -0.5770 & -0.5770 & -2.0195 & -0.8655\\1.4425 & 2.5965 & -0.5770 & -0.5770 & -0.8655 & -2.0195\\-0.5770 & -0.5770 & 0.5770 & 0.5770 & 0 & 0\\-0.5770 & -0.5770 & 0.5770 & 0.5770 & 0 & 0\\-2.0195 & -0.8655 & 0 & 0 & 2.0195 & 0.8655\\-0.8655 & -2.0195 & 0 & 0 & 0.8655 & 2.0195\end{bmatrix}\begin{array}{c}1x\\1y\\2y\\3x\\2x\\3y\end{array}$$

The global force vector is

$$\begin{bmatrix}r_{1x}\\r_{1y}\\r_{2y}+22.5\\r_{3x}+22.5\\22.5\\22.5\end{bmatrix}\begin{array}{c}1x\\1y\\2y\\3x\\2x\\3y\end{array}$$

Thus $\mathrm{K}_\mathrm{F}=10^{11}\begin{bmatrix}2.0195 & 0.8655\\0.8655 & 2.0195\end{bmatrix}$; $\mathrm{d}_\mathrm{F}=\begin{bmatrix}u_{2x}\\u_{3y}\end{bmatrix}$, $\mathrm{f}_\mathrm{F}=\begin{bmatrix}22.5\\22.5\end{bmatrix}$.

Since $d_E = 0$,

$$d_F = K_F^{-1} f_F = 10^{-10}\begin{bmatrix} 0.7799 \\ 0.7799 \end{bmatrix}.$$

Thus, $d^{(1)} = 10^{-10}\begin{bmatrix} 0 \\ 0 \\ 0.7799 \\ 0 \\ 0 \\ 0.7799 \end{bmatrix}$; $\varepsilon^{(1)}(1.5,1.5) = B^{(1)}d = 10^{-10}\begin{bmatrix} 0.2600 \\ 0.2600 \\ 0 \end{bmatrix}$

$$\sigma^{(1)}(1.5,1.5) = D^{(1)}\varepsilon^{(1)}(1.5,1.5) = \begin{bmatrix} 15 \\ 15 \\ 0 \end{bmatrix}.$$

Problem 2

For the two-dimensional loaded plate shown in Fig. 2, determine the displacement of nodes 1 and 2 and the element stresses using plane stress conditions. Body force may be neglected in comparison with the external forces.

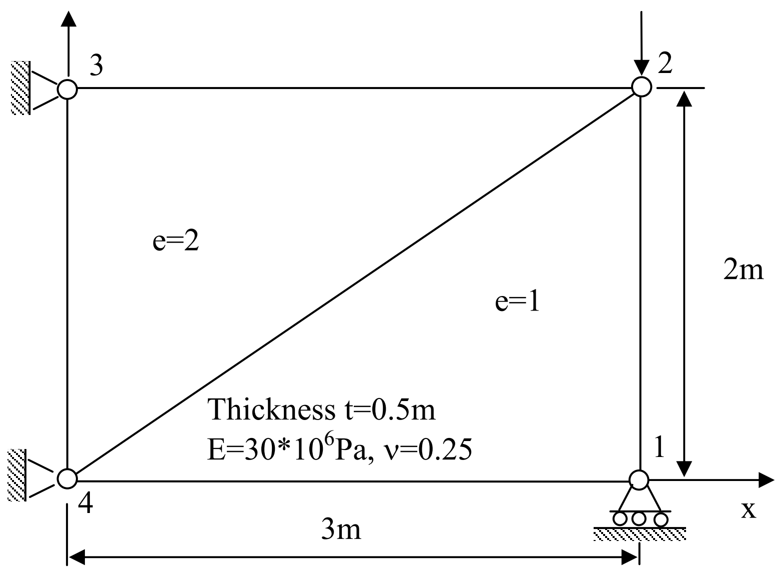

Fig. 2

Solution

As shown in Fig.2, for elment 1 (node 1, 2, 3), $x_1^{(1)} = 3$, $y_1^{(1)} = 0$, $x_2^{(1)} = 3$, $y_2^{(1)} = 2$, $x_3^{(1)} = 0$, and $y_3^{(1)} = 0$. For element 2 (node 3, 4, 2), $x_1^{(2)} = 0$, $y_1^{(2)} = 2$, $x_2^{(2)} = 0$, $y_2^{(2)} = 0$, $x_3^{(2)} = 3$, and $y_3^{(2)} = 2$.

With the plane stress condition, the constitutive matrix is

$$\mathrm{D}^{(1)} = \mathrm{D}^{(2)} = \frac{E}{1-\upsilon^2}\begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & (1-\upsilon)/2 \end{bmatrix} = 10^7 \begin{bmatrix} 3.2 & 0.8 & 0 \\ 0.8 & 3.2 & 0 \\ 0 & 0 & 1.2 \end{bmatrix}$$

Following the similar steps, for element 1, the strain-displacement matrix is

$$\mathrm{B}^{(1)} = \begin{bmatrix} 0.333 & 0 & 0 & 0 & -0.333 & 0 \\ 0 & -0.5 & 0 & 0.5 & 0 & 0 \\ -0.5 & 0.333 & 0.5 & 0 & 0 & -0.333 \end{bmatrix}.$$

For element 2, the strain-displacement matrix is

$$B^{(2)}=\begin{bmatrix} -0.333 & 0 & 0 & 0 & 0.333 & 0 \\ 0 & 0.5 & 0 & -0.5 & 0 & 0 \\ 0.5 & -0.333 & -0.5 & 0 & 0 & 0.333 \end{bmatrix}.$$

For element 1, the stiff matrix $K^{(1)} = t^{(1)}A^{(1)}B^{(1)T}D^{(1)}B^{(1)}$

$$=10^7 \begin{array}{c} \begin{array}{cccccc} 1x & 1y & 2x & 2y & 4x & 4y \end{array} \\ \begin{bmatrix} 0.983 & -0.5 & -0.45 & 0.2 & -0.533 & 0.3 \\ & 1.4 & 0.3 & -1.2 & 0.2 & -0.2 \\ & & 0.45 & 0 & 0 & -0.3 \\ & & & 1.2 & -0.2 & 0 \\ \mathit{Symmetry} & & & & 0.533 & 0 \\ & & & & 0 & 0.2 \end{bmatrix} \end{array} \begin{array}{c} \\ 1x \\ 1y \\ 2x \\ 2y \\ 4x \\ 4y \end{array}.$$

For element 2, the stiff matrix $K^{(2)}\ t^{(2)}A^{(2)}B^{(2)T}D^{(2)}B^{(2)}$

$$=10^7 \begin{array}{c} \begin{array}{cccccc} 3x & 3y & 4x & 4y & 2x & 2y \end{array} \\ \begin{bmatrix} 0.983 & -0.5 & -0.45 & 0.2 & -0.533 & 0.3 \\ & 1.4 & 0.3 & -1.2 & 0.2 & -0.2 \\ & & 0.45 & 0 & 0 & -0.3 \\ & & & 1.2 & -0.2 & 0 \\ \mathit{Symmetry} & & & & 0.533 & 0 \\ & & & & 0 & 0.2 \end{bmatrix} \end{array} \begin{array}{c} \\ 3x \\ 3y \\ 4x \\ 4y \\ 2x \\ 2y \end{array}.$$

The displacement vector is $d=\begin{bmatrix} u_{1x} \\ 0 \\ u_{2x} \\ u_{2y} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$

The force vector is $f_\Gamma + r=\begin{bmatrix} 0 \\ r_{1y} \\ 0 \\ -1000 \\ r_{3x} \\ r_{3y} \\ r_{4x} \\ r_{4y} \end{bmatrix}.$

Thus $\mathrm{d}_\mathrm{E} = 0$, $\mathrm{d}_\mathrm{F} = \begin{bmatrix} u_{1x} \\ u_{2x} \\ u_{2y} \end{bmatrix}$,

$$\mathrm{K}_\mathrm{F} = 10^7 \begin{array}{c} \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} \text{1x} & \text{2x} & \text{2y} \end{array} \\ \begin{bmatrix} 0.983 & -0.45 & 0.2 \\ -0.45 & 0.983 & 0 \\ 0.2 & 0 & 1.4 \end{bmatrix} \end{array} \begin{array}{c} \\ 1x \\ 2x \\ 2y \end{array},$$

and $\mathrm{f}_\mathrm{F} = \begin{bmatrix} 0 \\ 0 \\ -1000 \end{bmatrix}$.

Solving for u_{1x}, u_{2x}, u_{2y}, we obtain

$u_{1x} = 1.913 \times 10^{-5}$ m;

$u_{2x} = 0.875 \times 10^{-5}$ m;

$u_{2y} = -7.436 \times 10^{-5}$ m.

For element 1,

$$\mathrm{d}^{(1)} = 10^{-5} \begin{bmatrix} 1.913 \\ 0 \\ 0.875 \\ -7.436 \\ 0 \\ 0 \end{bmatrix}, \ \varepsilon^{(1)} = \mathrm{B}^{(1)}\mathrm{d}^{(1)} = 10^{-5} \begin{bmatrix} 0.638 \\ -3.718 \\ -0.519 \end{bmatrix}, \ \sigma^{(1)} = \mathrm{D}^{(1)}\varepsilon^{(1)} = \begin{bmatrix} -93.3 \\ -1138.7 \\ -62.3 \end{bmatrix}.$$

For element 2,

$$\mathrm{d}^{(2)} = 10^{-5} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.875 \\ -7.436 \end{bmatrix}, \ \varepsilon^{(2)} = \mathrm{B}^{(2)}\mathrm{d}^{(2)} = 10^{-5} \begin{bmatrix} 0.292 \\ 0 \\ 2.479 \end{bmatrix}, \ \sigma^{(1)} = \mathrm{D}^{(1)}\varepsilon^{(1)} = \begin{bmatrix} 93.4 \\ 23.4 \\ -297.4 \end{bmatrix}.$$