ES128: Homework 6
Solutions

Problem 1

Calculate the mass matrix for the two-dimensional square element shown in Fig. 1. The element has uniform thickness t and uniform density ρ.

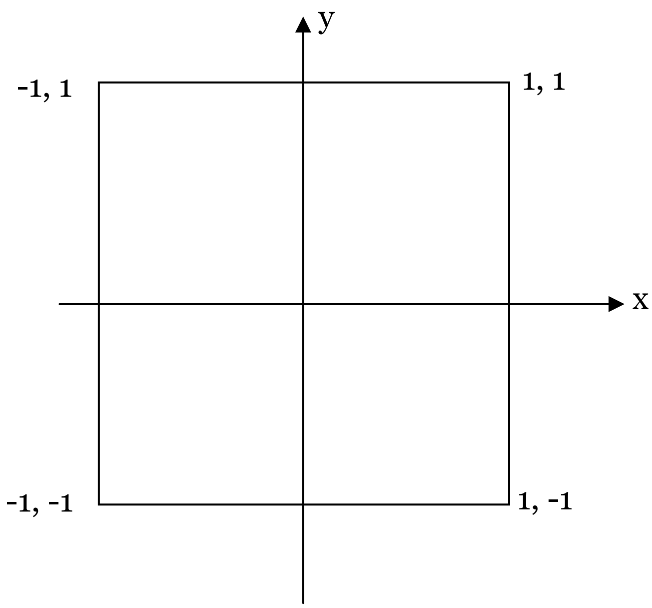

Fig.1

Solution

The shape functions are $N_1(x,y)=\frac{1}{4}(1-x)(1-y)$, $N_2(x,y)=\frac{1}{4}(1+x)(1-y)$,

$N_3(x,y)=\frac{1}{4}(1+x)(1+y)$, and $N_4(x,y)=\frac{1}{4}(1-x)(1+y)$.

The mass matrix is $\iiint_V [N]^T[N]\rho dV = \rho t\int_{-1}^{1}\int_{-1}^{1}[N]^T[N]dxdy$

$$= \rho t\begin{bmatrix} 4/9 & 2/9 & 1/9 & 2/9 & 0 & 0 & 0 & 0 \\ 2/9 & 4/9 & 2/9 & 1/9 & 0 & 0 & 0 & 0 \\ 1/9 & 2/9 & 4/9 & 2/9 & 0 & 0 & 0 & 0 \\ 2/9 & 1/9 & 2/9 & 4/9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4/9 & 2/9 & 1/9 & 2/9 \\ 0 & 0 & 0 & 0 & 2/9 & 4/9 & 2/9 & 1/9 \\ 0 & 0 & 0 & 0 & 1/9 & 2/9 & 4/9 & 2/9 \\ 0 & 0 & 0 & 0 & 2/9 & 1/9 & 2/9 & 4/9 \end{bmatrix}.$$

Problem 2

The four node parallelogram element shown below has uniform density and thickness. By integration, determine the consistent mass matrix (Hint: use the results of Problem 1 and the isoparametric formulation).

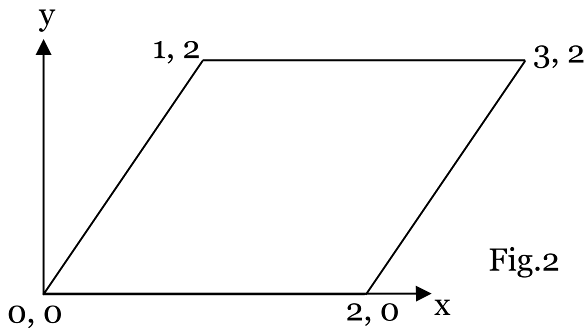

Fig.2

Solution

The shape functions in the parent domain are

$$N_1(r,s)=\frac{1}{4}(1-r)(1-s),$$

$$N_2(r,s)=\frac{1}{4}(1+r)(1-s),$$

$$N_3(r,s)=\frac{1}{4}(1+r)(1+s),$$

$$N_4(r,s)=\frac{1}{4}(1-r)(1+s).$$

The gradient in the parent domain is

$$\text{GN}=\begin{bmatrix} \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial r} & \frac{\partial N_4}{\partial r} \\ \frac{\partial N_1}{\partial s} & \frac{\partial N_2}{\partial s} & \frac{\partial N_3}{\partial s} & \frac{\partial N_4}{\partial s} \end{bmatrix}=\frac{1}{4}\begin{bmatrix} s-1 & 1-s & 1+s & -s-1 \\ r-1 & -r-1 & 1+r & 1-r \end{bmatrix}.$$

The element coordinate matrix is $[x\ y]=\begin{bmatrix} 0 & 0 \\ 2 & 0 \\ 3 & 2 \\ 1 & 2 \end{bmatrix}$.

The Jacobian matrix is $J=(GN)[x\ y]=\begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix}$, and $\det(J)=1$.

The mass matrix is given by $\iiint_V [N]^T[N]\rho dV = \rho t \int_{-1}^{1}\int_{-1}^{1}[N]^T[N]\det \mathrm{J} dxdy$

$$= \rho t \begin{bmatrix} 4/9 & 2/9 & 1/9 & 2/9 & 0 & 0 & 0 & 0 \\ 2/9 & 4/9 & 2/9 & 1/9 & 0 & 0 & 0 & 0 \\ 1/9 & 2/9 & 4/9 & 2/9 & 0 & 0 & 0 & 0 \\ 2/9 & 1/9 & 2/9 & 4/9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4/9 & 2/9 & 1/9 & 2/9 \\ 0 & 0 & 0 & 0 & 2/9 & 4/9 & 2/9 & 1/9 \\ 0 & 0 & 0 & 0 & 1/9 & 2/9 & 4/9 & 2/9 \\ 0 & 0 & 0 & 0 & 2/9 & 1/9 & 2/9 & 4/9 \end{bmatrix}.$$

Problem 3

Cross-sectional area of the bar (as shown in Fig. 3) varies linearly from A_0 at the left end to γA_0 at the right end, where γ is a constant. Determine the consistent mass matrix that operates on axial degree of freedom u_1 and u_2.

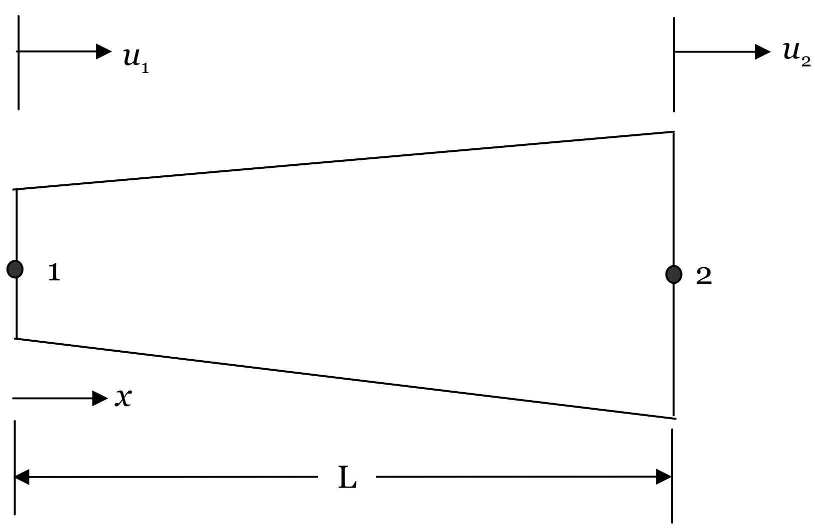

Fig. 3

Solution

The shape functions are

$$N_1(x) = \frac{L-x}{L},$$

$$N_2(x) = \frac{x}{L}.$$

The function of cross-sectional area is $A_0 + (\gamma - 1)A_0 \dfrac{x}{L}$.

The mass matrix is m= $\rho\int_V N^T N dV = \rho\int_0^L N^T N\left(A_0 + (\gamma-1)A_0\frac{x}{L}\right)dx$

$$= \frac{\rho A_0 L}{12}\begin{bmatrix}\gamma+3 & \gamma+1\\ \gamma+1 & 3\gamma+1\end{bmatrix}.$$

Problem 4

Only axial motion is permitted in the system shown in Fig. 4. Let k=1 and m=2. Determine the fundamental vibration frequency ω_1 of the given system. Then calculate ω_1 after condensing the system to a single degree of freedom using Guyan reduction.

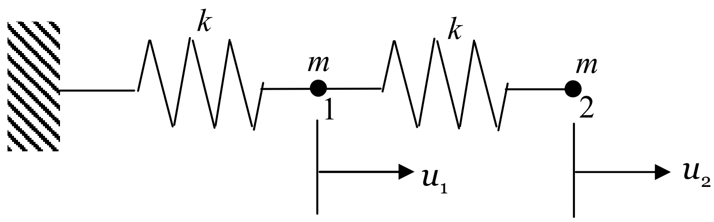

Fig. 4

Solution

The global stiffness matrix is $\begin{bmatrix}k & -k & 0\\ -k & 2k & -k\\ 0 & -k & k\end{bmatrix}$.

The global mass matrix is $\begin{bmatrix}0 & 0 & 0\\ 0 & m & 0\\ 0 & 0 & m\end{bmatrix}$.

The left end is fixed, so the equations of motion can be expressed as

$$\begin{bmatrix}m & 0\\ 0 & m\end{bmatrix}\begin{bmatrix}\ddot{u}_1\\ \ddot{u}_2\end{bmatrix}+\begin{bmatrix}2k & -k\\ -k & k\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}=0\,.$$

Let

$$\begin{vmatrix}2k-\omega^2 m & -k\\ -k & k-\omega^2 m\end{vmatrix}=0\,,$$

where ω is the natural frequency, and we can obtain $\omega_1 = 0.6180\sqrt{k/m}$ =0.437 and $\omega_2 = 1.6180\sqrt{k/m}$ =1.1441 .

Use Guyan reduction, let node 2 the master node, and eliminate the degree of freedom of node 1, so $K_{mm}=[k]$; $K_{ms}=[-k]$; $K_{sm}=[-k]$; $K_{ss}=[2k]$.

The transformation matrix

$$T = \begin{bmatrix} I \\ -K_{ss}^{-1} K_{sm} \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}.$$

The reduced stiffness maxtrix is $\tilde{K} = T^T K T = \begin{bmatrix} 1 & 0.5 \end{bmatrix} \begin{bmatrix} k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = 0.5k$

The reduced mass maxtrix is $\tilde{M} = T^T M T = \begin{bmatrix} 1 & 0.5 \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = 1.25m.$

Let $\tilde{K} - \omega^2 \tilde{M} = 0$, we obtain $\omega = \frac{1}{\sqrt{2.5}} \sqrt{\frac{k}{m}} = 0.4472$, which is close to 0.437.

Problem 5

A particle of unit mass is supported by a spring of unit stiffness, so 1 ω =1. There is no damping. At time t=0, when the particle has zero displacement and zero velocity, a unit force is applied and maintained. Use the central difference method to calculate displacement versus time over successive time steps as follows

1) Use Δt=0.5 and go to t=7
2) Use Δt=1 and go to t=7
3) Use Δt=2 and go to t=10
4) Use Δt=3 and go to t=15

Solution

Using the central difference method, we can obtain the function $d_{n+1} = f(d_n, d_{n-1})$.

At t=0, d_0=0 and $\dot{d}_0$=0. Based on $\dot{d}_0 = \frac{d_0 - d_{-1}}{\Delta t}$, we can obtain d_{-1}=0.

With $d_{n+1} = f(d_n, d_{n-1})$, we can get $d_1, d_2 \dots$ Fig. 5 shows the results for cases 1, 2, 3, and 4. As we can see, we can get the more accurate results by decreasing Δt.

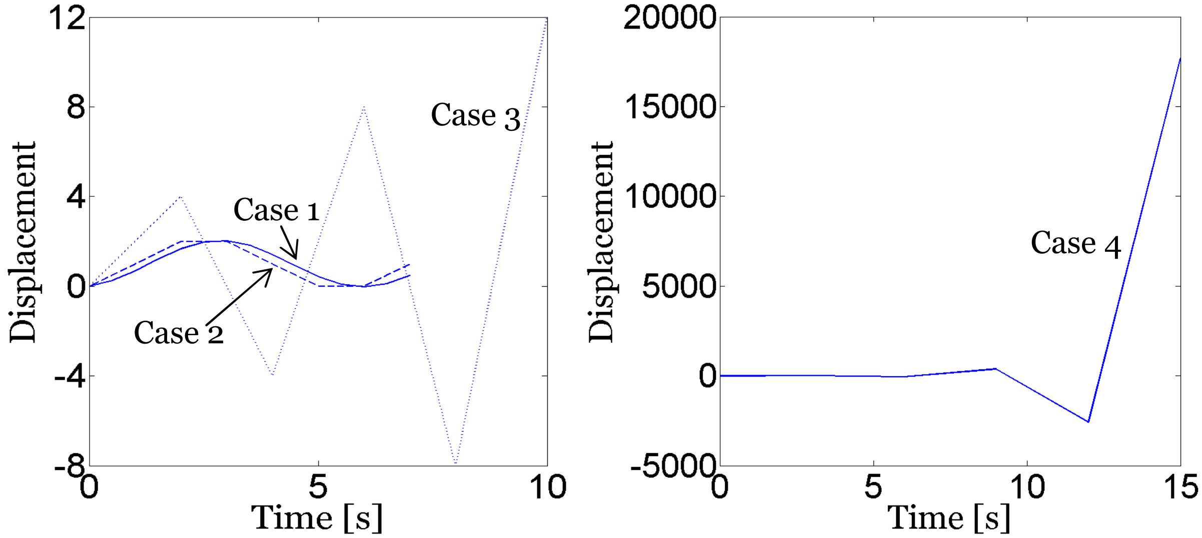

Fig. 5