APPLIED MECHANICS REVIEWS

Postbuckling Theory

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Introduction

A division of elastic stability theory and its applications into separate categories of buckling and postbuckling is not entirely rational, but the creation of such an artificial distinction is almost essential to bring either into manageable proportions. This point is brought home by more than 1600 references on cylindrical shell buckling alone in the survey by Grigol'uk and Kabanov [Ref. 1].

Postbuckling aspects of stability theory for flat plates received prominence in the early 1930's, after Wagner [Ref. 2] had established a sound theoretical foundation for the load-carrying capacity of deeply wrinkled flat shear panels. The approximate discussion by Cox for plates in uniaxial compression [Ref. 3] was followed by a rigorous analysis by Marguerre and Trefftz [Ref. 4]. This work was continued first in Germany [Refs. 5, 6], and somewhat later in the United States [Refs. 7-15], Britain [Refs. 16-18] and The Netherlands [Refs. 19-28]. Reviews of this work and subsequent developments are to be found in [Ref. 29]. The entirely different postbuckling behavior of thin shell structures came to light in the early 1940's when Karman and Tsien [Refs. 30-33] showed that the large discrepancies between test and theory for the buckling of certain types of thin shell structures was due to the highly unstable postbuckling behavior of these structures. At roughly the same time in war-time Holland, Koiter [Ref. 34] developed a general theory of stability for elastic systems subject to conservative loading which was published as his doctoral thesis in 1945.

Karman's and Tsien's papers spawned many more in the following 30 years, and most of them have been directed to obtaining more accurate results for the cylindrical shell under axial compression or to studying this structure under other loadings. Koiter's work, on the other hand, attracted relatively little attention until the early 1960's when interest in the general theory sprang up almost simultaneously in England and in the United States.

Our review will be centered mainly on the theory and applications that have emerged from these two approaches to postbuckling theory. A major stimulus to development in this subject continues to be generated by the quest for predictive methods for the buckling of shell structures, and a sizable portion of this review will concern applications of the theory to this area. A state-of-the-art discussion of shell buckling as of 1958 is given in the excellent survey by Fung and Sechler [Ref. 35], and that article provides a background for assessing the progress which has taken place in this subject in just over a decade. Our lack of command of the Russian language prevents us from doing justice to the contributions of Soviet scientists in this field. Reference may be made, however, to Volmir's outstanding treatise on the stability of deformable systems [Ref. 36], and to the survey on cylindrical shell buckling [Ref. 1].

Our discussion will not touch on research into stability of elastic systems subject to nonconservative loadings, but fortunately we can cite a recent survey in *Applied Mechanics Reviews* by Herrmann [Ref. 37] on this aspect of stability theory, as well as a similar survey by Nemat-Nasser [Ref. 38] on thermoelastic stability under general loads. We also would call attention to another recent review article by Thompson [Ref. 39] which does cover some of the same ground as the present survey, but the emphasis there is on elastic systems characterized by a finite number of degrees of freedom, whereas most of the applications we will discuss will be within the realm of continuum theory. We also omit a detailed discussion of secondary branching [Refs. 29, 40, 41] of the initially stable postbuckling behavior of flat plates, even if this phenomenon is of some importance for these and similar "completely symmetric" structures [Ref. 42].

Theory and Applications

The energy criterion of stability for elastic systems subject to conservative loading is almost universally adopted by workers in the field of structural stability [Refs. 43-45]. A positive definite second variation of the potential energy about a static equilibrium state is accepted as a sufficient condition for the stability of that state. Numerous attempts to undermine these two pillars of structural stability theory have been made, but confidence in them remains undiminished. With a proper shoring-up of certain aspects of these criteria, they will undoubtedly continue to serve as the foundation of elastic stability theory. An account of the present status of the fundamentals of stability theory is given in [Ref. 46].

In this review, we are concerned mainly with the class of problems most frequently encountered in the field of structural stability in which the loss of stability of one set of equilibrium states of an idealized, or "perfect," elastic structure is associated with bifurcation into another set of equilibrium states. The first set is referred to as the prebuckling state and the bifurcated state as the buckled configuration. The bifurcation load of the perfect structure is commonly called the classical buckling load and is denoted here by P_c. This is by no means the only circumstance under which a structure can become unstable. For example, a very shallow clamped spherical cap subject to a uniform pressure loading reaches a maximum, or limit, load at which point it becomes unstable under prescribed pressure with no occurrence of bifurcation. However, due to the symmetry of both the idealized structure and its loading, bifurcations frequently do occur at load levels below the limit load.

The classical analysis of the stability of the prebuckling state of the perfect structure takes the form of an eigenvalue problem for the lowest load level for which the second variation of the potential energy is semidefinite. The Euler equations associated with this variational principle are linear and the eigenmode (or modes) associated with the critical eigenvalue P_c is termed the buckling mode (or modes). While the classical analysis yields the load at which a perfect realization of a structure will at least start to undergo buckling deflections, it gives no indication of the character of the postbuckling behavior, nor does it give any insight into the way a nonperfect realization will behave.

Typical load-deflection curves characterizing static equilibrium configurations are shown in Fig. 1 for the case of structures which have a unique buckling mode associated with the classical buckling load. In each of the three cases shown the prebuckling state of the perfect structure is stable for $P < P_c$ and is unstable for $P > P_c$ where it is shown as a dotted curve. Case I illustrates a structure with a stable postbuckling behavior which can support loads in excess of P_c in the buckled state. The behavior of a slightly imperfect version of the same structure (or perhaps for a slightly misaligned application of the load) is depicted by the dashed curve. Case II is an example of a structure which goes into a stable or unstable postbuckling behavior depending on whether the load increases or decreases following bifurcation. An initial imperfection is all that is needed to prejudice the deflection one way or the other. If an imperfection induces a positive buckling deflection, the load-deflection curve of Case II has a limit load P_s, the buckling load of the imperfect structure, which is less than the bifurcation load of the perfect structure P_c. Case II is an example of asymmetric branching behavior, while Case III illustrates a structure whose buckling behavior is symmetric with respect to the buckling deflection and whose initial postbuckling behavior is always unstable under prescribed loading conditions.

The classical buckling load is a reasonably good measure of the load level at which an imperfect realization of the structure begins to undergo significant buckling deflections if the structure has a fully stable initial postbuckling behavior. In the early days, reliance on the classical buckling load stemmed from the fact that all confrontations between test and theory were for either columns or plates, and both of these are stable in the postbuckling regime. When tests were carried out on thin shell structures in the early 1900's, the classical theory became suspect because then actual buckling loads were frequently found to be as little as one-quarter of the classical load.

Except for a very few structures, such as the column under axial compression [Ref. 47] and the circular ring subject to inward pressure [Ref. 48], it is not possible to obtain closed form solutions governing the entire postbuckling behavior. Starting with Karman's and Tsien's work on spherical and cylindrical shells, a large number of numerical calculations in the large-deflection range have been made. Each of these calculations represents an attempt to obtain the load-deflection behavior of the perfect structure, and in particular the minimum load the structure can support in the buckled state (i.e., P_m in Fig. 1). This load was held to be significant as a possible design load on the grounds that the structure could always support at least this much load and that even imperfections would not reduce the buckling load below this value. This concept could not be useful in any universal sense since there are well-known examples of structures with negative minimum postbuckling loads. Another difficulty with this proposal is that the buckling process of such a structure is a dynamic one and the buckled structure may end up deformed quite differently from what is predicted on the basis of static equilibrium considerations alone. In any case, efforts to validate the minimum load for design purposes for cylindrical shell structures have not paid off, as we will discuss in the next major section of this article; and this idea perhaps should be abandoned.

Emphasis in the analysis of imperfection-sensitive structures has shifted to the maximum load P_s which can be supported before buckling is triggered and to relating P_s to the magnitude and forms imperfections actually take. This was the point of view adopted in the general nonlinear theory of stability [Ref. 34] and in subsequent investigations based on the general theory [Refs. 49-55].

The initial postbuckling analysis in its simplest form yields an asymptotically exact relation between the load parameter P and the buckling deflection δ, valid in the neighborhood of the bifurcation point of the perfect structure. This expansion is in the form

$$P = P_c(1 + a\delta + b\delta^2 + \dots) \tag{1}$$

where the coefficients a, b, ... determine the initial postbuckling behavior. A number of important problems are characterized by multiple buckling modes associated with the classical buckling load. Two of these problems will be discussed in the next section. For such problems, the initial postbuckling analysis may be considerably more complicated.

Asymptotically exact estimates of the buckling load of the imperfect structure are obtained by including the first-order effects of small initial deflections. Only the component of the initial deflection in the shape of the classical buckling mode enters into the resultant formula.

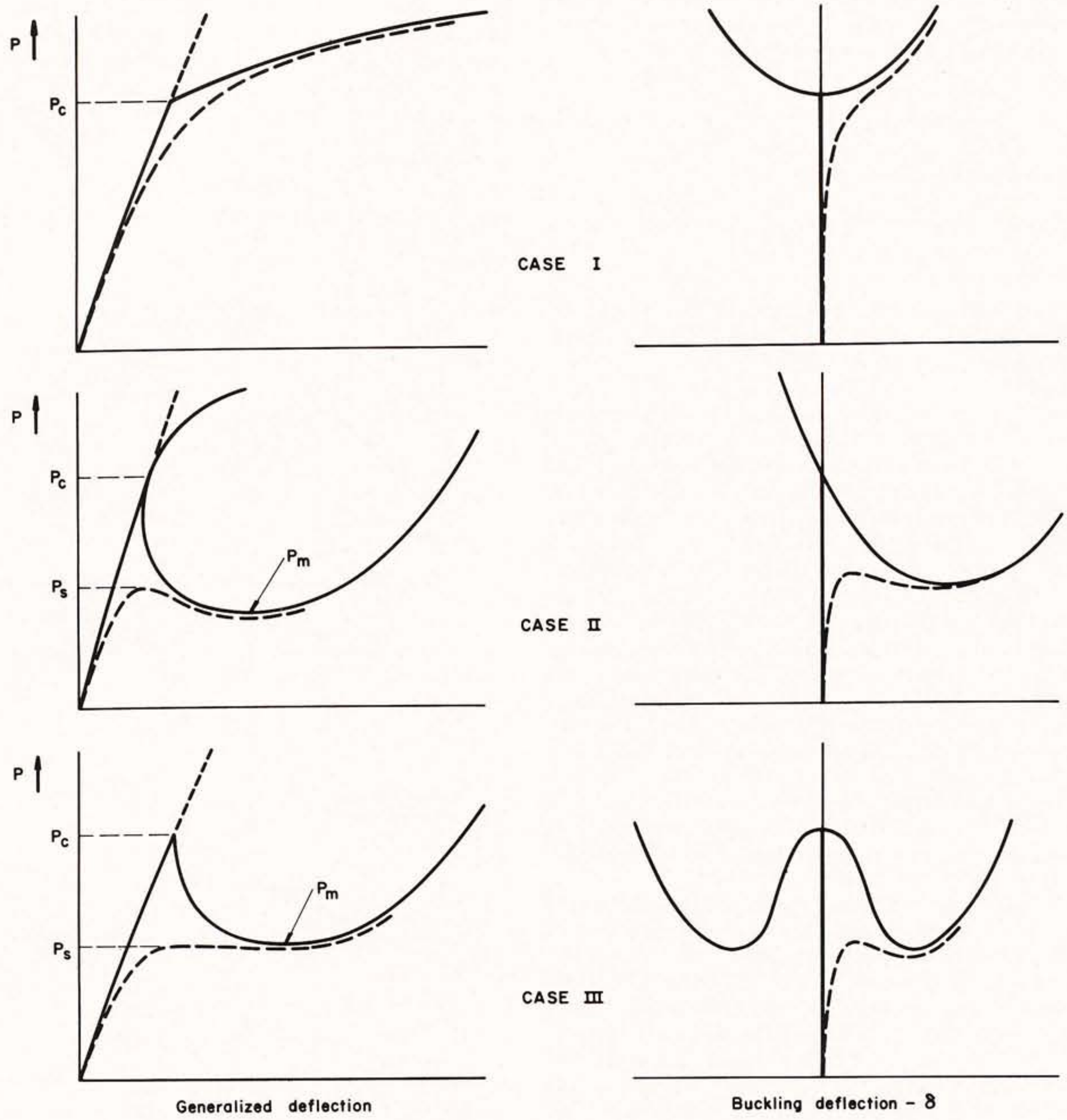

FIG. 1. LOAD-DEFLECTION CURVES FOR SINGLE MODE BIFURCATION BEHAVIOR

If the amplitude of this imperfection component is denoted by $\bar{\delta}$, then for Case II with $\bar{\delta} > 0$, the general theory yields

$$\left[1 - \frac{P_s}{P_c}\right]^2 = \alpha\bar{\delta}\ \frac{P_s}{P_c} \tag{2}$$

or

$$1 - \frac{P_s}{P_c} \sim (\alpha\bar{\delta})^{1/2}$$

where α is a constant which depends on properties of the structure. For the symmetric branching point of Case III ($a = 0$, $b < 0$), the analogous formula is

$$\left[1 - \frac{P_s}{P_c}\right]^{3/2} = \alpha|\bar{\delta}|\frac{P_s}{P_c} \tag{3}$$

or

$$1 - \frac{P_s}{P_c} \sim (\alpha|\bar{\delta}|)^{2/3} \quad .$$

In both instances, very small values of $\alpha\bar{\delta}$ have a sizable effect on P_s/P_c, and it is this feature which accounts for the extreme imperfection-sensitivity of many structures which have an unstable postbuckling behavior.

Interest in the initial postbuckling approach mushroomed in England in the early 1960's [Refs. 56-93]. Some fine experimentation is included in this work, and an extensive series of theoretical investigations have centered around conservative systems characterized by a finite number of degrees of freedom. A great deal of this work has been carried out at University College, London, by Chilver, Thompson, Walker and their students. Sewell, at the University of Reading, also has concentrated on the postbuckling behavior of discrete systems. He has particularly emphasized the systematics of the perturbation procedure and has used the terminology "the static perturbation method" in referring to this method of analysis. Both Thompson and Sewell have explored a variety of bifurcation possibilities for such systems. Recent studies have been made with an eye to providing a framework for postbuckling calculations via finite element methods. Some of this work is summarized in [Ref. 39].

Perhaps the best direct experimental validation of the initial postbuckling approach was obtained in a series of tests by Roorda [Refs. 67, 68]. The simple two-bar frame shown in Fig. 2 was loaded in a testing machine which, in effect, prescribed displacement rather than dead load and in this way the postbuckling equilibrium states were recorded. Experimental points and theoretical predictions [Refs. 53, 85] shown as a solid line for the load, P, versus buckling rotation, θ, for the "perfect" structure are displayed in the upper half of the figure. The agreement between theory and experiment for the effect of an imperfection – a slightly offset load – on the buckling load is remarkably good.

The first application of initial postbuckling theory was to the monocoque cylindrical shell under axial compression [Refs. 34, 50] which will be discussed in the next section. The second was a study of a narrow cylindrical panel under axial compression [Ref. 49] which displays a transition, depending on its width, from a stable postbuckling behavior typical of a flat plate to the unstable behavior which characterizes the cylindrical shell. Beaty [Ref. 94] and Thompson [Ref. 63] applied the general theory to the axisymmetric buckling of a

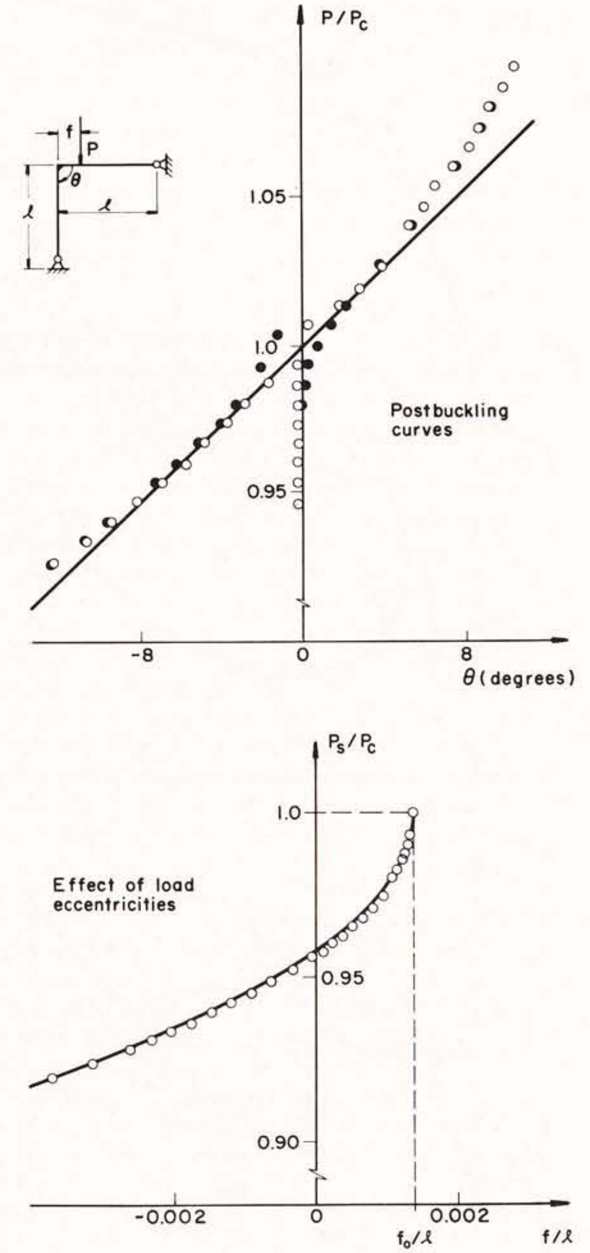

FIG. 2. COMPARISON OF THEORY AND EXPERIMENTS ON A SIMPLE TWO-BAR FRAME

complete spherical shell. Their analysis yielded the expected result that the sphere has an unstable postbuckling behavior. However, it has been shown recently [Ref. 55] that the initial postbuckling analysis in its simplest form leads to results that are of little practical consequence for the complete sphere problem. We defer a discussion of this matter also to the next section.

The general theory [Ref. 34] has been applied to a variety of shell structures by Budiansky and Hutchinson and their students at Harvard University [Refs. 95-110]. Included among them are toroidal shell segments under various loadings, cylindrical shells subject to torsion and spheroidal shells loaded by external pressure.

Results shown in Fig. 3 due to Budiansky and Amazigo [Ref. 103] very nicely illustrate the outcome of such an analysis and its interpretation. In the upper half of the figure the classical buckling pressure p_c in nondimensional form is plotted as a function of the length parameter Z appropriate for either a simply-supported cylinder of length L or a segment of length L of an infinite cylinder reinforced by rings which permit no lateral deflection but allow rotation. The initial postbuckling behavior is symmetric with respect to the buckling amplitude δ, and therefore the pressure-deflection relation takes the form

$$\frac{p}{p_c} = 1 + b\left(\frac{\delta}{t}\right)^2 + \dots \tag{4}$$

where b is plotted in the lower half of Fig. 3. In the figure, D is the bending stiffness, ν is the Poisson ratio and t is the shell thickness. In this case, the asymptotic relationship between the buckling pressure and the imperfection is

$$\left[1 - \frac{p_s}{p_c}\right]^{3/2} = \frac{3\sqrt{3}}{2} \; (-b)^{1/2} \left|\frac{\bar{\delta}}{t}\right| \frac{p_s}{p_c} \tag{5}$$

where $\bar{\delta}$ is the amplitude of the component of the imperfection in the shape of the classical buckling mode. A wide range of test data, collected by Dow [Ref. 111], is also included in the figure. Measurements of initial deflections were not made in any of these tests, so it is not possible to make a direct comparison of test and theory. On the other hand, the coincidence of the large discrepancy between test and classical predictions within the Z-range in which b is most negative bears out the imperfection-sensitivity predicted.

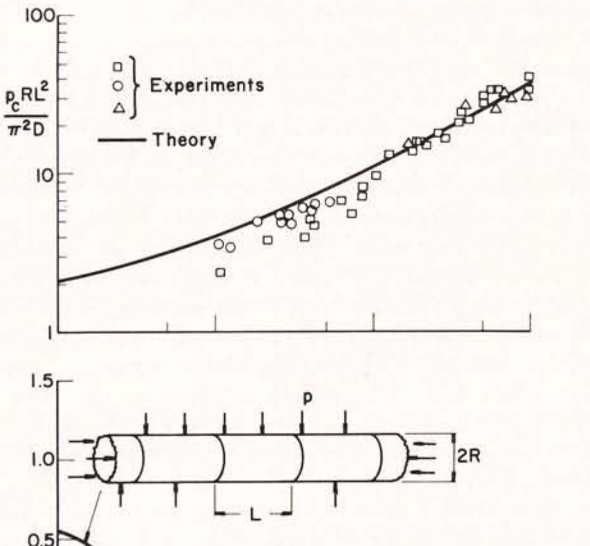

FIG. 3. COMPARISON BETWEEN TEST AND CLASSICAL THEORY WITH INITIAL POSTBUCKLING PREDICTIONS FOR A CYLINDRICAL SHELL UNDER EXTERNAL PRESSURE

The complementary character of an initial postbuckling analysis and a large deflection analysis is brought out by studies of long oval cylindrical shells under axial compression. Kempner's and Chen's [Refs. 112-114] numerical results for the advanced postbuckling behavior of shells with sufficiently eccentric oval cross section showed that loads above the classical buckling load could be supported. Complete collapse occurs only when the buckling deflections engulf the high curvature ends of the shell. According to an initial postbuckling analysis [Ref. 105], initial buckling will be just about as sensitive to imperfections as for the circular cylindrical shell as would indeed be expected because of the shallow buckling behavior in each case. The composite picture is that initial buckling will be imperfection-sensitive but not catastrophic, and loads above the classical values should be possible. Recent tests [Ref. 115] have confirmed both these features.

One aspect of shell buckling which has attracted a great deal of theoretical and experimental attention in the last few years is the role which stiffening plays in strengthening shell structures. Some rather unexpected effects have turned up. One of the most interesting was the observation by van der Neut [Ref. 116] over 20 years ago that the axial buckling load of a longitudinally stiffened cylindrical shell can be increased significantly by attaching the stiffeners to the outer surface of the shell rather than the inner surface. This advantage has been clearly demonstrated by tests [Ref. 117].

Initial postbuckling results [Refs. 102, 109, 118] indicate that stiffened cylindrical shells tend to be less imperfection-sensitive than their unstiffened counterparts, but the level of sensitivity can vary widely. For example, while the classical buckling load of an outside-stiffened cylinder may be much higher (a factor of two is not untypical) than an inside-stiffened one, the imperfection-sensitivity of the outside-stiffened shell may be much higher as well. Only a few experiments are available to put these predictions to a test but those which have been reported [Refs. 119, 120] show precisely this trend.

An extensive series of buckling tests on stiffened shell structures has been underway for a number of years under the direction of Singer at the Technion in Israel [Refs. 121, 122]. This program has produced ample evidence that the classical buckling load is a more reliable measure of actual buckling loads for stiffened shells than it is for unstiffened shells. Nevertheless, the experimental scatter and the discrepancies from the classical predictions in these tests are indicative of the fact that stiffening does not eliminate the problem of imperfection-sensitivity. An up-to-date discussion of this matter is given in [Ref. 123].

Other recent work in the area of postbuckling theory includes an investigation of the interaction of local buckling and column failure for thin-walled compression members [Refs. 124, 125]. This work may help to eradicate the naive approach to optimal design of structures liable to buckling by an attempt to equalize the local and overall buckling stresses.

There have been important advances in both theoretical and calculational aspects of shell buckling in the last decade. Buckling equations have been proposed [Refs. 52, 126] which are exact within the context of first-order shell theory. Computer programs are now available for accurate computation of classical buckling loads for a wide class of shells of revolution subject to axially symmetric loads [Refs. 127, 128]. These programs incorporate effects of nonlinear prebuckling behavior and discrete rings. When the prebuckling deformation of the perfect shell is a purely membrane one with no bending, the initial postbuckling analysis is generally simpler than when bending, and nonlinear prebuckling effects must be taken into account. Most applications of postbuckling theory to date have been in cases in which the prebuckling response is exactly a purely membrane state or could be reasonably approximated by one. Within the last three years there have been several applications of the general theory to problems in which it is essential to include nonlinear prebuckling effects [Refs. 104, 109, 110, 129], and a general purpose computer program for shells of revolution subject to axisymmetric loads has been put together [Ref. 130].

Status of the Postbuckling Theory of the Cylindrical Shell under Axial Compression and the Spherical Shell under Uniform Pressure

The cylindrical shell under axial compression has served as the prototype in studies of shell buckling [Refs. 131-161], but its geometric simplicity is deceptive, and in many respects this structure, together with the sphere subject to pressure, has the most complicated postbuckling behavior of all. In large part, this stems from the fact that a large number of buckling modes are associated with the classical buckling load in each of these problems, and consequently these shells are susceptible to a wide spectrum of imperfection shapes.

It is quite possible that a paper by Hoff, Madsen and Mayers [Ref. 152] has put an end to the quest for the minimum load which the buckled shell can support. A sequence of large deflection calculations of the minimum postbuckling load have been reported, each calculation more accurate than those which preceded it. Hoff, et al., give a convincing argument that a completely accurate calculation, based on the procedure which had been employed in all the previous investigations, would lead to a value for the minimum postbuckling load which would tend to zero for a vanishing thickness to radius ratio. Of course, a shell with a finite thickness to radius ratio would actually have a nonzero minimum load, but there now appears to be general agreement that this minimum is not nearly as relevant as had been thought.

The role of boundary conditions was extensively explored in the 1960's and is now fairly well settled. Accurate numerical calculations of classical buckling loads which account for nonlinear prebuckling effects and various boundary conditions have been made [Ref. 162]. The classical load for cylindrical shell of moderate length which is clamped at both ends is about 93 per cent of the load predicted by the well-known "classical" formula based on a calculation which ignores end conditions altogether. So-called weak boundary conditions for which no tangential shear stress is exerted on the ends of the shell reduce the buckling load by a factor of about two [Refs. 163, 164]. More recently it has been shown that relaxed tangential restraint along a small fraction of the edge has an almost equally detrimental effect [Ref. 165].

Near perfect cylindrical shell specimens have been produced by Tennyson [Ref. 166] and Babcock and Sechler [Ref. 144], and these shells buckle very close to the classical buckling load. High-speed photography has resolved the apparent discrepancy between observed buckling patterns and the classical mode shapes [Refs. 167-171]. Buckle wavelengths associated with the collapsed shell are much longer than the classical buckling mode wavelengths, but the wavelengths associated with deformation patterns photographed just after buckling has been triggered are in good agreement with the predictions of initial postbuckling theory.

The long cylindrical shell subject to axial compression is one of the examples used in [Ref. 34] to illustrate the general theory of elastic stability. Boundary conditions are neglected, and consideration is restricted to mode shapes which are periodic in both the axial and circumferential coordinates. Due to the large number of simultaneous buckling modes, there are many possible static equilibrium branches which emanate from the bifurcation point of a perfect cylinder, and all of them are unstable. Remarkably, the initial curvature of the curve of load versus end shortening (P versus ϵ) is the same for all of them and just after bifurcation

$$\frac{\epsilon}{\epsilon_c} = \frac{P}{P_c} + \frac{2}{3}\left[\frac{P}{P_c} - 1\right]^2 . \qquad (6)$$

Effects of certain imperfections were also studied. Among the shape possibilities represented by all the classical buckling modes, a geometric imperfection in the shape of the axisymmetric buckling mode results in the largest relative reduction in the buckling load [Ref. 51]. The ratio of the buckling load to the classical load is related to the amplitude of such an axisymmetric imperfection $\bar{\delta}$ by

$$\left[1 - \frac{P_s}{P_c}\right]^2 = \frac{3c}{2}\left|\frac{\bar{\delta}}{t}\right|\frac{P_s}{P_c} \qquad (7)$$

where $c = \sqrt{3(1-\nu^2)}$ and ν is Poisson ratio. When the maximum load P_s is attained, the shell has undergone lateral deflections which are only a fraction of the shell thickness, and an imperfection amplitude of only one-fifth a shell thickness causes almost a 50 per cent reduction in the buckling load.

Eq. (7) is an asymptotic formula; and like all initial postbuckling predictions of this type, it is accurate only for sufficiently small imperfection amplitudes. In general, it is difficult to assess the range of validity of asymptotic results. For this particular imperfection shape, it is possible to obtain an independent and more accurate estimate of the P_s versus $\bar{\delta}$ relation which is based on a finite deflection calculation [Ref. 51]. The asymptotic predictions (eq. 7) are plotted with the more accurate results in Fig. 4, and two predictions compare well over a substantial part of the range of interest. The calculation of [Ref. 51] was repeated by Almroth [Ref. 172] with the aim of getting better accuracy. He found that the buckling load prediction may be somewhat lower than those shown in Fig. 4 if the parameter Z (Fig. 3) is very large.

Some experimental verification of these buckling load calculations has been obtained by Tennyson and Muggeridge [Ref. 173] with tests of cylindrical shell specimens with axisymmetric sinusoidal imperfections which were carefully machined into them. Points representing the experimental loads for five shells are plotted with two theoretical curves in Fig. 5. The solid curve is the result of the finite deflection prediction of [Ref. 51] for the imperfection wavelength coincident with those of the specimens. The dashed curve is a plot of the results of a numerical calculation which takes into account nonlinear prebuckling deformations associated with the clamped end conditions and the thickness variations present in the test cylinders [Ref. 173]. At extremely small imperfection levels, end conditions dominate. Once the imperfection amplitude is just a few per cent of the shell thickness, the end effect is negligible and the two predictions are virtually identical.

There have been some efforts to translate and adapt the initial postbuckling predictions into a form directly useful for engineering design [Refs. 129, 130, 161].

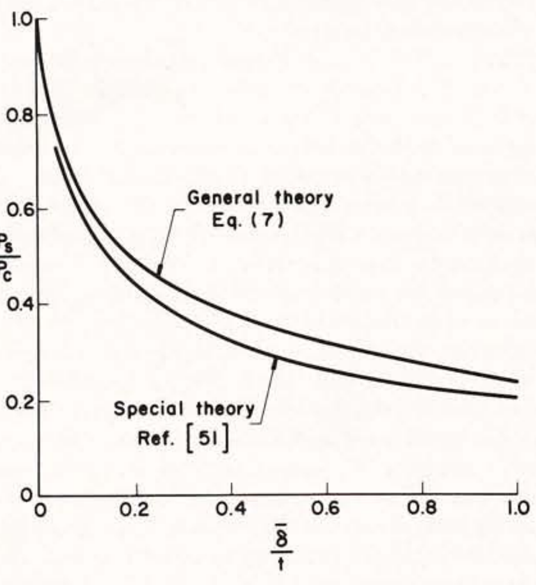

FIG. 4. EFFECT OF AXISYMMETRIC IMPERFECTIONS IN THE SHAPE OF THE AXISYMMETRIC BUCKLING MODE OF A PERFECT CYLINDRICAL SHELL UNDER AXIAL COMPRESSION

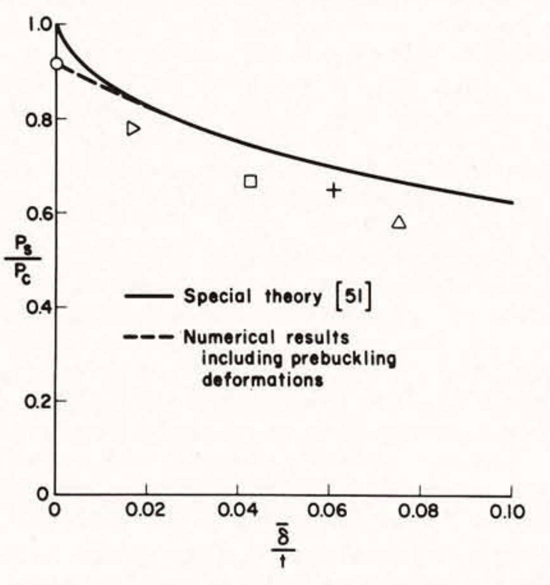

FIG. 5. COMPARISON BETWEEN TEST AND THEORY FOR BUCKLING OF CYLINDRICAL SHELLS WITH AXISYMMETRIC IMPERFECTIONS LOADED IN AXIAL COMPRESSION

Reinforcing this trend is recent work by Arbocz and Babcock [Refs. 174, 175] in which imperfections present in a number of test specimens were carefully measured and mapped. Special techniques have been used to calculate buckling loads for certain imperfection shapes which are quite different from the classical buckling modes or for imperfection shapes which are only characterized statistically. Some of these will be touched on in the next section.

Since Karman's and Tsien's [Ref. 30] first large deflection calculation for the axisymmetric postbuckling behavior of a complete spherical shell under uniform pressure, a fairly large number of papers on the buckling of caps and spheres have been published. Many of these are referenced in a recent paper on the subject [Ref. 55]. Here, we will only mention some of the salient features peculiar to the sphere problem which have recently come to light.

Like the cylindrical shell under axial compression, a complete spherical shell under external pressure has many buckling modes associated with the classical buckling pressure, and one of these is axisymmetric. Many unstable equilibrium branches emanate from the bifurcation point of the perfect shell. A shallow shell analysis of multimode buckling of the sphere [Ref. 101] indicates that this shell is highly sensitive to both axisymmetric and nonaxisymmetric imperfections, again just as for the cylinder. While this shallow shell analysis is not rigorously applicable to the complete sphere, its relative simplicity reveals the mechanics of the buckling phenomena of the sphere very clearly. In particular, the shallow buckling modes are in striking similarity with experimental modes reported in [Ref. 176]. An alternative approach to this problem by a straightforward perturbation technique is developed in [Refs. 177, 178].

Initial postbuckling analyses for axisymmetric buckling have been given by Beaty [Ref. 94], Thompson [Ref. 63], Walker [Ref. 80], and most recently by Koiter [Ref. 55]. A multimode analysis in which the axisymmetric mode couples with nonaxisymmetric modes, analogous to cylindrical shell behavior, also was given in [Ref. 55]. Imperfection-sensitivity was found to be higher for multimode buckling than for axisymmetric buckling.

A most important discovery of [Ref. 55] was that the axisymmetric results just mentioned, which were obtained by a perturbation expansion about the bifurcation point, have an extremely small range of validity which actually vanishes as the thickness to radius ratio of the shell goes to zero. The principal reason for this unusual limitation on the initial postbuckling results is the occurrence of modes, associated with eigenvalues which are only very slightly larger than the critical eigenvalues, whose amplitudes become comparable in magnitude to the amplitudes of the classical buckling modes outside of a very small neighborhood of the bifurcation point. For all practical purposes, the initial postbuckling analysis in its standard form breaks down under such circumstances. A more powerful form of asymptotic expansion, also first detailed in [Ref. 34], with an extended range of validity was applied to the axisymmetric problem [Ref. 55]. The outcome of this special analysis is that the spherical shell is highly imperfection-sensitive, more or less to the same degree as the cylindrical shell under axial compression, even when the deformations are constrained to be axisymmetric. An approximate analysis [Ref. 179] and several numerical calculations [Refs. 180-183] back up this conclusion.

New Research Directions

It seems safe to predict that much of the impetus to postbuckling theory as a developing subject will continue to come from questions which arise in the analysis of shell structures. Efforts are being directed to interpreting postbuckling predictions and rendering them useful for engineering purposes. If this is to be accomplished, further calculations will be needed for more realistic imperfections than those which are readily accommodated by the initial postbuckling analysis. Approaches which incorporate statistical descriptions of imperfections are also likely to receive growing attention in the coming years.

Very recently, Sewell [Ref. 184] has explored various ramifications of the more powerful expansion method of [Ref. 34] (which was employed to get around the difficulties in the sphere problem referred to above) as it applies to conservative systems with a finite number of degrees of freedom. Thompson [Ref. 185] has proposed a somewhat different variation of the method, also for discrete rather than continuous systems. The aim in each case is to develop a uniformly valid asymptotic expansion which gives an extended range of validity for those rather exceptional problems in which the standard expansion breaks down outside the immediate vicinity of the bifurcation point.

Effects of localized dimple imperfections on the buckling of a beam on a nonlinear elastic foundation have been explored using a variety of techniques, and asymptotic formulas analogous to eqs. (2) and (3) have been uncovered [Ref. 186]. Application of these special techniques has been made to study the effect of local nonaxisymmetric imperfections on buckling of cylindrical shells under external pressure [Ref. 187]. In addition, both theoretical and experimental studies have been made of cylindrical and conical shells under axial compression in the presence of axisymmetric dimple imperfections [Refs. 188, 189]. With a continuing rapid growth of numerical methods of shell analysis, it should soon be possible to make reasonably accurate calculations of the buckling load of an arbitrary shell structure in the presence of almost any form of imperfection. The potential of such methods has already been demonstrated [Refs. 190-193] for problems in which it is necessary to convert the governing nonlinear partial differential equations to an algebraic system by spreading a two-dimensional finite-difference grid over the shell middle surface. No doubt calculations of this sort will not be inexpensive in terms of computation time for many years to come; but if they are carefully selected, such calculations should be most useful for imperfection-sensitivity assessment.

The development of numerical methods for carrying out the initial postbuckling analysis seems likely to progress along the lines of both the finite difference procedure [Ref. 130] and the finite element method [Refs. 80, 194]. An interesting scheme has been put forward for approximately accounting for nonlinear prebuckling behavior in an initial postbuckling analysis in a way which exploits the computational advantages of the finite element procedure [Refs. 195, 196]. The idea behind the scheme is the treatment of the prebuckling nonlinearity as a generalized imperfection.

Preliminary attempts have been made to come to grips with some of the statistical aspects of imperfection-sensitivity [Refs. 197-199]. A rather complete analysis [Ref. 200] of an infinite beam with random initial deflections resting on a nonlinear elastic foundation yields a relationship between the buckling load of the beam and the root mean square of the imperfection amplitudes with an implicit dependence on the correlation function for the imperfection. Calculations for the buckling load of an infinite cylindrical shell under axial compression with random axisymmetric imperfections have been made [Refs. 201, 202]. Here again, there emerge asymptotic formulas similar to eqs. (2) and (3) which now relate the buckling load to a value of the imperfection power spectral density at a particular frequency which corresponds to the frequency of the classical axisymmetric mode [Ref. 201]. There is still a long way to go before results such as these will be useful to the structural engineer, but an encouraging first step in this direction has been taken [Ref. 203].

Important extensions of postbuckling theory remain to be made to include buckling under dynamically applied loads and buckling in the plastic range. Dynamic buckling calculations are not new, but no unifying theory is available which is at all as comprehensive as that for static conservative loadings. The initial postbuckling analysis has been broadened in an approximate fashion to include dynamic effects [Refs. 95-97], and some simple results with general implications have been found.

Buckling in the plastic range is a phenomenon that is not yet fully understood. Nonuniqueness of incremental solutions in plastic branching problems starts at the lowest bifurcation load, usually without loss of stability, and a further load increase is required initially in the postbuckling range [Refs. 204, 205]. A detailed discussion of the famous column controversy among Considere, Engesser, von Karman and Shanley [Refs. 206-211] is given in [Ref. 212]. In the case of plates and shells the problem is even more complicated because the physically unacceptable deformation theory of plasticity yields results in better agreement with experiments than the simplest flow theory. Even if flow theory is certainly more acceptable from the physical point of view, it is somewhat doubtful, however, that the simplest J_2 theory would be adequate in buckling problems. To some extent this discrepancy also has been explained by Onat and Drucker [Ref. 213], by proper allowance for initial imperfections in their simple example. Even more significant, perhaps, are Hutchinson's recent results on the postbuckling behavior in the plastic range of structures which are highly imperfection-sensitive in the elastic domain [Ref. 214]. It is found in his examples (a simple structural model and a spherical shell under external pressure) that the sensitivity to imperfections is equally marked in the plastic range, in spite of the *initial* rise in the load at the lowest bifurcation point of the perfect structure. A similar phenomenon has been observed by Leckie [Ref. 215] in the postcollapse behavior of a rigid-plastic spherical shell cap. Hutchinson's analysis also confirms the earlier findings [Ref. 213] that the discrepancy between the predictions of deformation theory and flow theory largely disappears in the presence of imperfections.

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