Equilibrium of an elastically confined liquid drop

Hyuk-Min Kwon,¹ Ho-Young Kim,¹,a) Jérôme Puëll,² and L. Mahadevan²,b)
¹School of Mechanical and Aerospace Engineering, Seoul National University, Seoul 151-744, Republic of Korea
²School of Engineering and Applied Sciences, Harvard University, Cambridge, Massachusetts 02138, USA

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When a liquid drop is confined between an elastic plate and a rigid substrate, it spreads spontaneously due to the effects of interfacial forces, eventually reaching an equilibrium shape determined by the balance between elastic and capillary effects. We provide an analytical theory for the static shape of the sheet and the extent of liquid spreading and show that our experiments are quantitatively consistent with the theory. The theory is relevant for the first step of painting when a brush is brought down on to canvas. More mundanely, it allows us to understand the stiction of microcantilevers to wafer substrates occurring in microelectromechanical fabrication processes.


I. INTRODUCTION

The first and often crucial step in Chinese calligraphy and certain forms of painting such as pointillism is the bringing of a wet brush or other soft paint-bearing element to the canvas. This leads to a spontaneous spreading of a drop of paint due to the effects of interfacial forces. Here we analyze this simple process to determine the shape of the brush and the drop as a function of the various parameters that characterize the problem. This study complements our previous work on the capillary rise of a liquid between two elastic plates, a problem motivated by the uptake of paint by a soft elastic brush.¹² However, there is a qualitative difference between the elastocapillary rise problem treated in Ref. 2 and that considered here since volume conservation rather than gravity determines the final equilibrium shape of the drop and plate. This type of problem also arises in some technological situations associated with microelectromechanical systems where a common process involves thin microstructures such as cantilever beams or bridges that can be fabricated by a surface micromachining process called wet release etch. In this process, a sacrificial layer deposited on a substrate, over which a structural material is deposited and patterned, is removed by wet etching to leave an overhanging structure. Upon post-etch rinsing, surface tension pulls in the compliant beams as the rinsing solution dries. If the bending stiffness of the beam is small (in a way that we shall quantify), it is brought into contact with the substrate and they may adhere firmly together, a phenomenon generally called stiction.³ Indeed, this process is a simple instance that shows how the combination of interfacial forces and nonequilibrium kinetics such as that induced by drying can lead to the self-assembly of soft simple elements into complex structures.

Here we focus on a minimal system consisting of a drop that lies between two plates of length, L, and width, w, separated by a distance, H, which is much smaller than L and w. If the drop volume \( \tilde{\Omega} \gg H^2 \), then volume conservation implies that it spreads into an elliptical drop having the length of approximately \( \tilde{\Omega} / Hw \) with deviations from the shape arising due to the capillary effects along the edge of the drop. If one or both plates are flexible and hydrophilic, capillary forces associated with the meniscus curvature lead to a negative pressure in the liquid causing the gap between the sheets to be modulated. For a relatively short and stiff sheet, the liquid spreads slightly more due to this effect and is accompanied by a slight decrease in the gap between the sheets. However, when the sheets are long and flexible, they can deform substantially and even stick to each other leading to a qualitatively different behavior. To understand these at a simple level, we start with the relatively stiff regime. When a sheet of length, L, thickness, t, density, \( \rho_s \), and bending stiffness, B, is deflected through a distance, \( \delta \), due to surface tension, \( \sigma \), balancing the torque exerted on the sheet, \( B \delta / L^2 \), with the capillary torque, \( \sigma \tilde{\Omega} (H - \delta)^2 \), yields \( \delta \sim \sigma \tilde{\Omega} L^3 / BH^2 \), where the two-dimensional volume, \( \tilde{\Omega} = \tilde{\Omega} / w \). Denoting the extent of spreading by \( l_m \), volume conservation gives \( \tilde{\Omega} \sim (H - \delta) l_m \) so that \( l_m = l_r (1 - k / \eta)^{-1} \), where \( l_r = \tilde{\Omega} / H \). Here \( \eta = (l_r / L)^2 (H / L)^2 (HL / \tilde{\Omega}) \) is a parameter that characterizes the ratio of elastic and capillary forces up to geometric factors, and \( l_r = (B / \sigma)^{1/2} \) is the adhesion or bending length.⁴ The constant \( k \) is of order unity, showing how the effects of plate stiffness appear perturbatively to modify the extent of drop spreading. We note that as \( \eta \rightarrow \infty \), we recover \( l_r \sim l_r \). As \( \eta \) decreases, on the other hand, the effects of capillary adhesion become important so that the sheet deforms more, leading to a higher degree of spreading. When \( \eta \ll 1 \), the sheet eventually sticks to the rigid plate starting from its free end, so that the quantity of interest is the dry length \( l_d \). Minimizing the sum of the elastic energy of the deformed sheet \( BH^2 / l_r^2 \) and the interfacial energy \( \sigma l_r (l_r - l_d) \) yields \( l_d \sim (B / \sigma)^{1/4} H^{1/2} \sim (l_r H)^{1/2} \). In the following, we formulate and solve a free boundary problem to understand the quantitative dependence of liquid spreading and sheet shape on the various problem parameters to complement these simple scaling estimates.
II. THE FREE BOUNDARY PROBLEM: FORMULATION, SOLUTION, AND EXPERIMENTAL CORROBORATION

We consider a liquid drop in equilibrium between a rigid horizontal flat plate and a flexible sheet clamped at an angle \( \alpha_0 \), as shown in Fig. 1. When the clamping distance \( H \equiv L \), we may use linear plate theory to describe its deformation \( h(x) \) (Ref. 5)

\[
Bh'''(x) = q(x),
\]

(1)

where \( B \) is the bending stiffness per unit width of the sheet, \( h' = dh/dx \), and \( q(x) \) is the force per unit area on the sheet.

A. Spreading under a stiff plate with a separated end

For \( \eta \approx O(1) \), i.e., in the relatively stiff regime, the sheet is in contact with liquid over an unknown length, \( l_m = L - x_m \), while its free end is still some distance from the bottom plate as shown in Fig. 1(a). We neglect the effect of gravity on both the drop and the elastic plate. The conditions for this are (i) the Bond number of the drop \( Bo = \rho g H^2/\sigma \ll 1 \) and (ii) the torque due to weight \( M_y = \rho g L^2 \) is small compared with the torque due to surface tension \( M_z \sim \sigma \Omega L^2/H^2 \), i.e., \( L \ll \Omega/\rho g H^2 \). When a water drop and the glass sheets used for the computation in Fig. 2 are considered, \( Bo \) is below 0.09 and the ratio \( M_z/M_y \) is of the order of 10\(^{-2}\). In micro-machining processes to fabricate 1 \( \mu m \)-thick silicon nitride beams overhanging 100 \( \mu m \) from the substrate, the numbers are even smaller, \( Bo \) and \( M_z/M_y \) being of the order of 10\(^{-3}\) and 10\(^{-5}\), respectively.

Then the relative pressure in the liquid is \(-\sigma/R_0\), where \( R_0 \) is the radius of curvature of the meniscus. Then the pressure distribution over the entire range of \( x \) can be simply written as \( q = -(\sigma/R_0)\Theta(x - x_m) \), where \( \Theta(.) \) is the Heaviside function. Substituting into Eq. (1) yields

\[
Bh''' = -\frac{\sigma}{R_0}\Theta(x - x_m),
\]

(2)

To complete the formulation of the problem, we need some boundary conditions. As the plate is clamped at \( x = 0 \), it follows that \( h(0) = H \) and \( h'(0) = \tan \alpha_0 \). At the other end, the sheet is free of torques but is subject to a transverse shear force due to surface tension so that \( h''(L) = 0 \) and \( Bh'''(L) = \sigma \sin \theta_t \), where \( \theta_t \) is the contact angle of the liquid with the end of the sheet. These four boundary conditions must be supplemented by matching conditions at the meniscus, \( x = x_m \), given by the continuity of the deflection, the slope, and the curvature of the sheet, i.e., \( [h] = [h'] = [h''] = 0 \), where \( [A] = \lim_{x \to -0}(A(x_m + e) - A(x_m - e)) \). However, there is a jump in the transverse shear force across \( x = x_m \) due to surface ten-

![Schematic of the free boundary problem](image)
upon dropping the hats, as

\[ \varphi = \theta_i - \alpha_m. \]

Here \( \alpha_m \) is the slope of the sheet at \( x = x_m \), thus \( \tan \alpha_m = h'(x_m) \). The right-hand meniscus is directly deduced from the shape of the left-hand one as the pressure in the drop is a constant when gravitational effects are neglected. The condition corresponding to the force due to surface tension and the sheet is soft and long (low \( l_s/L \)), while the separation at the clamped end is small (low \( H/L \)). Not surprisingly, a large amount of liquid (high \( \Omega/HL \)) with small contact angles spreads more. Lowering \( l_s/L \) and \( H/L \) and increasing \( \Omega/HL \) correspond to decreasing the dimensionless stiffness \( \eta \). As \( \eta \) decreases further, the sheet becomes relatively more flexible leading to contact with the bottom plate. This changes the problem qualitatively due to a change in the boundary conditions, a case treated in the next section.

To compare these results with experiments, we start with a glass cover slip cleaned with piranha solution to make them almost perfectly wettable by water and ethylene glycol (EG). The sheet width \( w = 4 \mathrm{mm} \) and the thickness \( t = 150 \mu \mathrm{m} \), respectively, and its length \( L \in [19.1 \, 38.7] \mathrm{mm} \). The drop volume \( \Omega \in [0.858 \, 13.74] \mu \mathrm{m}^3 \). For the bottom plates, we used ParaFilm M laboratory sealing film (PF: American National Can, Chicago, IL) having an equilibrium contact angle \( \theta_i = 89^\circ \) with EG and polycarbonate having the contact angle \( \theta_i = 87^\circ \) with water. Figure 3(a) shows that the scaled spreading length, \( l_s/L \), increases as the scaled height, \( H/L \), decreases, revealing good agreement between the theory and the experiments. For small gaps, i.e., low values of \( H/L \), liquid spreading is greatly enhanced for soft elastic plates, i.e., small \( \eta \), as compared with that under rigid plates. In Fig. 3(b), we compare the experimental measurements with the scaling introduced earlier, \( l_s \sim l_s(1 - k/\eta)^{-1} \), which is also reasonably good for small values of \( 1/\eta \).

**B. Spreading under a soft sheet with a contacting end**

If we increase the sheet length, or decrease its flexural rigidity or gap height to make \( \eta \approx 1 \), surface tension causes the sheet to touch the bottom plate at the free end. Figure 4 shows the change of sheet shapes on increasing the sheet length \( L \) while maintaining the gap height \( H \) and the liquid volume \( \Omega \). When the sheet length increases to, say, \( L_s \), the sheet touches the bottom, and the formulation of the resulting free boundary problem changes slightly since the boundary condition corresponding to the force due to surface tension on the right side of the sheet \( h''(1) = L^3 \sin \theta_i / H^2 \) is replaced by the kinematic condition \( h(1) = 0 \). As \( \eta \) becomes...
even smaller, the angle between the plates at their contacting
ends, still at \( x=L \), decreases until eventually it vanishes when
the sheet length is, say, \( L=L_2 \), leading to smooth tangential
contact between the plates along a line (line III in Fig. 4).
Still further increase in the sheet length, with \( L>L_2 \), then
causes the contact line between the sheets to move to an
unknown location \( x=x_c \). In this new regime, there are two
unknown locations: the wet-dry meniscus \( x_m \) and the location
of the contact line \( x_c \), beyond which the sheet and the bottom
plate are effectively in contact [see Fig. 1(b)]. When \( L>L_2 \),
the quantity of interest is the size of the dry and wet regions,
and the sheet shape for \( 0<x<x_c \) is independent of the
length of the entire sheet. The governing Eq. (5) still holds
for \( 0<x<x_c \) and the boundary conditions at \( x=0 \) and the
matching conditions at \( x=x_m \) are identical to the foregoing
formulations. At \( x=x_c \), \( h(x_c)=h'(x_c)=0 \) consistent with tan-
gentially smooth contact. Furthermore, assuming a thin inter-
calating layer of liquid even in the region \( x_c<x<L \), where
the solid sheets are in nominal contact, yields \( h''(x_c)=0 \).\(^2\)
Solving the differential Eq. (5) with the additional conditions
earlier gives the sheet shape and the two unknown locations
\( x_m \) and \( x_c \).

Figure 5(a) shows the experimental and theoretical re-
sults for the transition between the separated-end regime
and the contacting-end regime. As the sheet height decreases,
the separated-end solution ceases to be valid when the sheet
contacts the substrate, entering the contacting-end regime (path
A in the figure). On the other hand, when raising the sheet
from the end-contact configuration, we experimentally find
that the sheet does not detach from the substrate following
the path A. Instead the end-contacting regime persists until
the path B. This is because the shear force acting at the free
end of the sheet continues to be upward (negative \( \gamma_m \)) until
the sheet is separated from the substrate. If we consider the
additional contribution of surface tension acting along the
width of the drop, this accounts for the persistence of the
end-contacting configuration between paths A and B. Indeed
such a mechanism has been proposed qualitatively to explain
the release etch processing for microfabrication that is re-
sponsible for the stiction behavior of a microcantilever under
which a drying liquid drop exerts a surface tension force
toward the substrate.\(^6\) Adding the surface tension force acting
at both sides of the sheet (\( -2\sigma \)) divided by its width to \( q(x) \)
in Eq. (1), the right-hand side of Eq. (2) becomes
\( -\sigma(1/R_0+2/w)H(x-x_m) \). Here we have approximated the surface
tension at the sides as acting vertically. This yields the wet-
tend length as a function of \( H/L \) as shown by the dotted line
in Fig. 5(a). The shear force acting at the free end of the
sheet calculated using this modified model vanishes at the
point denoted as a cross. This agrees well with the experi-
mentally found transition point indicated by the path B. For
other ranges, including the surface tension acting at the sides
of the sheet changed values of \( l_m/l \) little (less than 5\%). We
note that the spreading length \( l_m \) increases with the height
decrease much more sensitively in the contacting-end regime
than in the separated-end regime.

Figure 5(b) shows reasonable agreement between exper-
iment and theory for the transition between the end-contact
regime and tangentially smooth-contact regime. The figure
also shows the aforementioned scaling law \( l_m/L=1-l/l_H \)
\( =1-k(l/L)^{1/2}/L \), with the adjustable constant \( k=1.7 \), which
agrees well with the experiments.

III. DISCUSSION

Our quantitative theory and experiments for the spreading
of an elastically confined drop has probed the simplest
elements of the configuration of the sheet and drop as a func-
tion of the fluid volume, the length of the sheet, and its
distance from the substrate as well as the stiffness of the
sheet and surface tension of the liquid. We have shown that
there are three distinct configurations of the sheet depending
on the value of the scaled stiffness of the sheet which com-
bines geometric and physical parameters. Our focus has been
on two-dimensional configurations; much remains to be done for more complex geometries associated with, say, the multiple hairs of a paint brush. This problem may serve as a precursor toward a qualitative theory of how more complex structures might arise from these simple interactions when modified by kinetic processes such as drying.

We now conclude by adding some discussions on how this problem can be extended to more complex but relevant situations. When the drop does not touch the free end initially but rather located between the clamped and the free ends, the drop will eventually move to the free end to reach the equilibrium. This is because the pressure in the liquid around the meniscus near the clamped end (having a smaller interface curvature) is higher than the pressure near the free end (having a larger interface curvature) thus propels the drop toward the free end. Hence, the current formulation can still be used for the situation considered. When the sheet width is reduced to a comparable size to the gap height \( w - R_0 \), the three-dimensional effect comes into play. A mere inclusion of the surface tension force acting at both sides of the sheet as discussed earlier to explain the dotted line in Fig. 5(a) can result in good estimates for the shapes of the sheet and the drop provided that the interface curvature along the side is neglected. Since the contact angle at the side is not simply defined due to the fact that the angle is now formed at the solid edge, considering the interface profile along the side involves a much more complex study. In general, for narrow sheets, additional effects of the surface tension forces that pull the sheet downward tend to enhance both the sheet deformation and the drop spreading. Similar three-dimensional effects are to be considered for circular filaments, a geometry of most real brushes, for which our present study can provide qualitative estimates for the two-dimensional filament deformation and the degree of liquid spreading.

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